Contents

1 Taylor Series

1.1 Problems

Problem 1.1: Compute the Taylor Series of the following functions to second order around $x = 0$.

(a) e^x

(b) $\sin x$

(c) $\cos x$

(d) $(1+x)^\alpha$ where α is any arbitrary real number. This is often used when α is $2, 3, \frac{1}{2}$, or $-\frac{1}{2}$.

(e) $\log(1+x)$

Problem 1.2: Special relativity extends classical mechanics so that it can describe objects traveling near the speed of light, $c = 3\times10^8\,\mathrm{m\,s^{-1}}$. In special relativity, the energy of a free object is $E = \gamma mc^2$ (this is related to the famous $E = mc^2$) and the momentum is $p = \gamma mc$; where

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}.$$

Show that if $v \ll c$, then these relativistic equations become the equations for energy and momentum you're familiar with from Physics 7A: $E = \frac{1}{2}mv^2$ and $p = mv$. For a 1000 kg car traveling at $25\,\mathrm{m\,s^{-1}}$ (about 60 miles per hour), approximately big is the special relativistic correction?

Problem 1.3: (More Difficult.) If $x^x = 0$ when $x = 1$, then show

$$\int_0^1 x^x\,dx = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^n}$$

by first finding the Taylor series for x^x and then integrating each term in the series separately.

1.2 Solutions

Solution 1.1: Compute the Taylor Series of the following functions to second order around $x = 0$.

(a) e^x

(b) $\sin x$

(c) $\cos x$

(d) $(1+x)^\alpha$ where α is any arbitrary real number. This is often used when α is $2, 3, \frac{1}{2}$, or $-\frac{1}{2}$.

(e) $\ln(1+x)$

Recall that the formula for computing Taylor series in general is

$$f(x) = \sum_{i=1}^{\infty}\frac{1}{n!}\left.\frac{d^n f}{dx^n}\right|_{x=x_0}(x-x_0)^n. \tag{1}$$

When we say "compute to second order" we mean to only take the first three terms in the sum — only up to $(x-x_0)^2$, and "around $x = 0$" means $x_0 = 0$.

$f(x)$	e^x	$\sin(x)$	$\cos(x)$	$(1+x)^\alpha$	$\ln(1+x)$
$f(0)$	1	0	1	1	0
$\frac{df}{dx}$	e^x	$\cos(x)$	$-\sin(x)$	$\alpha(1+x)^{\alpha-1}$	$\frac{1}{1+x}$
$\frac{df}{dx}\Big\vert_{x=0}$	1	1	0	α	1
$\frac{d^2f}{dx^2}$	e^x	$-\sin(x)$	$-\cos(x)$	$\alpha(\alpha-1)(1+x)^{\alpha-2}$	$-\frac{1}{(1+x)^2}$
$\frac{d^2f}{dx^2}\Big\vert_{x=0}$	1	0	-1	$\alpha(\alpha-1)$	-1

Therefore

$$e^x \approx 1 + x + \tfrac{1}{2}x^2 \tag{2}$$
$$\sin(x) \approx x \tag{3}$$
$$\cos(x) \approx 1 - \frac{x^2}{2} \tag{4}$$
$$(1+x)^\alpha \approx 1 + \alpha x + \tfrac{1}{2}\alpha(\alpha-1)x^2 \tag{5}$$
$$\ln(1+x) \approx x - \frac{x^2}{2} \tag{6}$$

These are the five most useful Taylor series and you will be using them frequently in this course. You will probably end up memorizing them by accident by the end of the semester.

Solution 1.2: Special relativity extends classical mechanics so that it can describe objects traveling near the speed of light, $c = 3 \times 10^8\,\mathrm{m\,s^{-1}}$. In special relativity, the energy of a free object is $E = \gamma mc^2$ (this is related to the famous $E = mc^2$) and the momentum is $p = \gamma mv$; where

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

Show that if $v \ll c$, then these relativistic equations become the equations for energy and momentum you're familiar with from Physics 7A: $E = \frac{1}{2}mv^2$ and $p = mv$. For a 1000 kg car traveling at $25\,\mathrm{m\,s^{-1}}$ (about 60 miles per hour), approximately big is the special relativistic correction?

If $v \ll c$, then $\frac{v}{c} \ll 1$, so it makes sense to do a Taylor series in the quantity v/c. Define $x = (v/c)^2$. Then

$$\gamma(x) = \frac{1}{\sqrt{1-x}} = (1-x)^{-1/2} = 1 + \frac{x}{2} + \frac{3x^2}{8} + \cdots \tag{7}$$

Therefore

$$E = \gamma(v)mc^2 \approx \left(1 + \frac{x}{2}\right)mc^2 = \left(1 + \frac{1}{2}\frac{v^2}{c^2}\right)mc^2 = mc^2 + \frac{1}{2}mv^2. \tag{8}$$

The first term here is Einstein's famous equation — that the energy stored in mass is mc^2 — and the second term is the kinetic energy you're familiar with from mechanics. Similarly,

$$p = \gamma mv = \left(1 + \frac{1}{2}\frac{v^2}{c^2}\right)mv = mv + \frac{1}{2}mv\frac{v^2}{c^2}. \tag{9}$$

The correction to the momentum is very small, and can usually be ignored.

Solution 1.3: (More Difficult.) If $x^x = 0$ when $x = 1$, then show

$$\int_0^1 x^x\,dx = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^n}$$

by first finding the Taylor series for x^x and then integrating each term in the series separately.

This is left as a challenge problem, since it's not very related to the material of this course. One may show that

$$x^x = 1 + (x-1) + (x-2)^2 + \frac{1}{2}(x-1)^3 + \frac{1}{3}(x-1)^4 + \cdots \tag{10}$$

and then integrate this term-by-term.

2 Thermal Expansion

2.1 Helpful Equations

$$\begin{aligned} L(T) &= L_0\left(1+\alpha\Delta T\right) \\ V(T) &= V_0\left(1+\beta\Delta T\right). \end{aligned}$$

2.2 Problems

Suppose we have a rod of length L at temperature T. As you probably know, objects usually expand when they heat up and usually contract when they cool down. We can then think of the length as the rod as a function of temperature: $L(T)$.

For most materials, however, these changes in temperature are small near room temperature, so we can use a linear model for $L(T)$ around room temperature T_0. From our knowledge of Taylor series:

$$L(T) \approx L\left(T_0\right) + \left.\frac{dL}{dT}\right|_{T=T_0} (T - T_0). \tag{11}$$

If we call the $L_0 = L(T_0)$ the initial length, then we can write this as

$$L(T) = L_0\left(1 + \frac{1}{L_0}\left.\frac{dL}{dT}\right|_{T=T_0}(T-T_0)\right) \implies L = L_0\left(1+\alpha\Delta T\right) \tag{12}$$

where $\Delta T = T - T_0$ and $\alpha = \frac{1}{L_0}\frac{dL}{dT}\big|_{T=T_0}$. Here α is called the **coefficient of linear expansion**, which is different for each material. A typical value is 10^{-5} (of some unspecified standard units) when using the Kelvin scale.

Problem 2.1: What are the units of α? Does the value of α change if you work in Kelvin vs. Fahrenheit? What about Kelvin vs. Celsius?.

Material	**Linear Expansion Coeff.** (α)
Aluminium	2.5×10^{-5}
Copper	1.7×10^{-5}
Iron/Steel	1.2×10^{-5}
Lead	2.9×10^{-5}
Glass	0.9×10^{-5}
Quartz	0.4×10^{-6}
Concrete	1.45×10^{-5}

Table 1: Coefficients of Linear Expansion for some common materials. Units are omitted so as not to spoil the answer to problem 1, but are compatible with the Kelvin system of temperature. *Source: Giancoli.*

Problem 2.2: A concrete highway is built of slabs 12 m long (at 15 °C). How wide should the expansion cracks between the slabs be (at 15 °C) to prevent buckling if the range of the temperature is −30 °C to 50 °C? *Source: Giancoli.*

Using a similar Taylor series to what we used above in Equation (11), we can model the volume expansion of an object by

$$\Delta V = V_0\left(1+\beta\Delta T\right) \tag{13}$$

where β is called the **coefficient of volume expansion**.

Problem 2.3: What are the units of β?

Once a Taylor series-type approximation has been performed, say in temperature, only terms up to some power in ΔT appear in our equations. This is well-justified because if ΔT is small, then $(\alpha\Delta T)^2$ is very very small indeed, and we can ignore it with only a very small error. In general, these expansions will only make sense if some dimensionless quantity is small. It doesn't make sense to say "1 meter is small", since 1 meter is small compared to the Berkeley stadium, but large compared to the tip of a pencil. One should only do a Taylor series if you have a small dimensionless quantity to expand in. Angles are a common choice here, because radians are always dimensionless. What is our dimensionless quantity in this case?

Physicists often keep only the linear term in their dimensionless quantity, perhaps x. This is called **expanding to first order in** x. When working to first order in x, then you can automatically write $x^2 = 0$, $x^3 = 0$, etc.

Problem 2.4: Consider a rectangular prism with length ℓ, width w and height h. Suppose its coefficient of linear expansion is α in each direction. Use this information to find the change in volume to first order in $\alpha\Delta T$. What is the relationship (write an equation) between α and β for the prism?

Problem 2.5: Now consider a sphere of radius r whose coefficient of linear expansion is α is in each direction. Use $V = \frac{4}{3}\pi r^3$ to determine the change in volume of the sphere to first order in ΔT. Is the relationship between α and β the same or different?

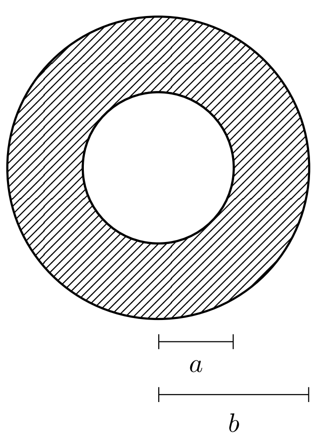
Problem 2.6: You have a rectangular prism that has a length L, a height H, and a width W: First, consider that the thermal expansion of the prism is isotropic (equal in all directions). It has a linear expansion coefficient α. What is the volume expansion coefficient of the prism for a small temperature change?

Now consider that the thermal expansion of this prism is anisotropic. Its length expands with linear expansion coefficient α_L, and its other dimensions expand with linear expansion coefficient α_{HW}: What is the volume expansion coefficient of the prism for a small temperature change? *Source:* Modified from 7B Supplement, p. 25

Problem 2.7: How can we tell that the formula for volume expansion works for any shape? Maybe we just got lucky — after all, rectangular prisms and spheres are about the nicest 3D shapes there are. Generalize you result from the previous two problems to *any* arbitrary 3D shape, even those without a formula. (The answer here is just a geometric explanaion, not a calculation.)

Problem 2.8: Suppose we have a disk of mass m and initial radius R_0. It is spinning about its perpendicular disk at angular velocity ω, and there is a motor in place to maintain this angular velocity. The temperature of the disk is raised linearly from T_1 to T_2 over a time t. What torque is required by the motor to keep it spinning at ω? You can assume that the disk has a coefficient of linear thermal expansion α.

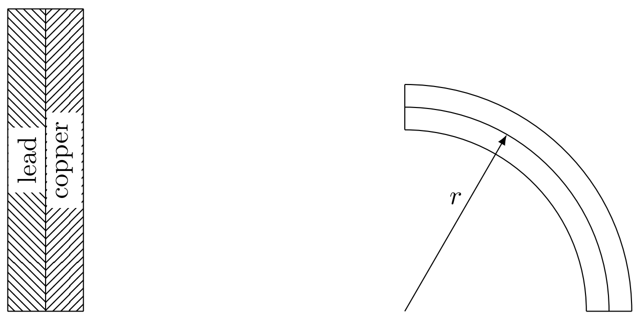
Problem 2.9: Suppose we have an annulus with inner radius a and outer radius b at temperature T_0. (See picture.) Does the radius of the inner circle change when we heat up the annulus? (*Hint*: first work out how thermal expansion works for a circle or for 2D objects in general.)



Problem 2.10: At a given latitude, ocean water in the so-called "mixed layer" (from the surface to a depth of about 50 m) is at approximately the same temperature due to the mixing action of waves. Assume that because of climate change, the temperature of the mixed layer is everywhere increased by 0.5 °C, while the temperature of the deeper portions of the oceans is unchanged. Estimate the resulting rise in sea level. The ocean covers about 70% of the Earth's surface, the radius of the Earth is approximately 6400 km, and $\beta_{\text{water}} = 2.10 \times 10^{-5}\text{K}^{-1}$. (*Source*: Giancoli.) Note: the answer is surprisingly small. Melting ice leads to much larger changes.

Problem 2.11: A bimetallic strip made of copper on one side and lead on the other. Each of the metals is 2.0 mm thick. At 20 °C, the strip is 10.0 cm long and straight. Find the radius of curvature r at 100 °C. How could you use this to make a toaster turn off at the right time?

Note: This problem is often on the homework, so GSIs should not explain how to do it.]



Problem 2.12:

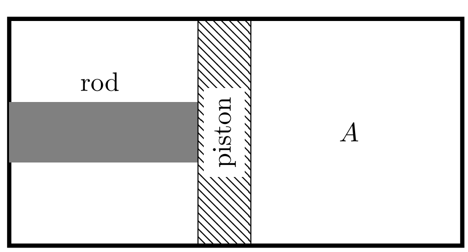
(a) Show that if the lengths of two rods of different solids are inversely proportional to their respective coefficients of linear expansion at some initial temperature, then the difference in length between them will be constant at all temperatures.

(b) What should the lengths of a steel and brass rod be at 0 °C so that their difference in length is 0.30 m. Use $\alpha_{\text{Steel}} = 1.2 \times 10^{-5}\,\text{K}^{-1}$ and $\alpha_{\text{Brass}} = 2 \times 10^{-5}\,\text{K}^{-1}$

Modified from: Page 542, Problem 8, Halliday & Resnick, 1966 Edition

Problem 2.13: Consider the system shown below which consists of an outer box, a compartment (labelled A on the diagram) filled with an ideal gas, a piston, and a rod. The piston serves as the left wall of A and can move left and right. A rigid rod connects the piston to the outer box, holding it in place.

The rod is made of a strange material: its coefficient of thermal expansion is $\alpha = 0.05\text{K}^{-1}$. Furthermore, the outer box has a length of 1 m, the rod has an initial length $L_0 = 0.4$ m, and the piston has a width of 0.1 m. The initial temperature of the entire system is $T_0 = 60$ °C. Suppose we increase the temperature of the *rod only* to $T_{\text{f}} = 65$ °C. By what factor does the pressure of the gas change? (Use the ideal gas law.) *Source: GSI Sachdeva.*



Problem 2.14: **Subject: Thermal Expansion** An alcohol thermometer is made of a cylindrical tube of inner diameter d_0 and a bulb of volume V_0 at room temperature T_0. The volumetric coefficients of thermal expansion as β_{al} and β_{g} for the alcohol and glass respectively, with $\beta_{\text{al}} \gg \beta_{\text{g}}$. Assume the thickness of the glass is negligible.

(a) If the volume of the bulb is much bigger than that of the tube, determine the change in volume of the inside of the thermometer when the temperature is increased from T_0 to T.

(b) Determine the change in volume of the alcohol between the same two temperatures.

(c) What is the change in height of the column of alcohol between T_0 and T?

(d) What is the change in the height of the column of alcohol between the same temperatures if the volume of the tube cannot be neglected?

Source: Bordel Midterm 1, Spring 2015. `https://tbp.berkeley.edu/courses/physics/7b/`

2.3 Solutions

Solution 2.1: The units of α are inverse temperature. This can be either $^\circ\mathrm{C}^{-1}$, K^{-1} or $^\circ\mathrm{F}^{-1}$. Since a *difference* of one degree Celsius is exactly the same as a difference of 1 Kelvin, the value of α is the same in Kelvin and Celsius, but different in Fahrenheit. No one should use Fahrenheit in scientific problems, and Celsius should be avoided as much as possible.

Solution 2.2: The slabs will buckle if they expand enough to touch. Therefore the slabs must be far enough apart so that they barely touch at 50 °C. Since the slabs expand on both ends, but there are two slabs expanding into each gap, the distance between slabs should be

$$d = L_0 \alpha \Delta T = \left(12 \cdot 1.45 \times 10^{-6} \cdot 35\right) \mathrm{m} = 0.609\,\mathrm{cm}.$$

Solution 2.3: The units on β are also inverse temperature.

Solution 2.4: The volume of the rectangular prism is $V = \ell w h$. To first order in $\alpha \Delta T$, each dimension will expand the same way:

$$\begin{aligned} \ell &= \ell_0(1+\alpha\Delta T), \\ w &= w_0(1+\alpha\Delta T), \\ h &= h_0(1+\alpha\Delta T). \end{aligned}$$

Therefore

$$\begin{aligned} V &= \ell_0(1+\alpha\Delta T)w_0(1+\alpha\Delta T)h_0(1+\alpha\Delta T) \\ &= \ell_0 w_0 h_0 \left[1 + 3\alpha\Delta T + 3\left(\alpha\Delta T\right)^2 + (\alpha\Delta T)^3\right] \\ &\approx V_0\left[1+3\alpha\Delta T\right]. \end{aligned}$$

In the last line here, we used the fact that we are working to first order in $\alpha\Delta T$ to set $\left(\alpha\Delta T\right)^2 = 0$ and $\left(\alpha\Delta T\right)^3 = 0$. This is, of course, not literally true. However, for most reasonable physical situations, $\alpha\Delta T \approx 10^{-6}$, so these quantities are tiny and can safely be ignored.

By comparison with the equation $V = V_0\left[1+\beta\Delta T\right]$, we find $\beta = 3\alpha$.

Solution 2.5: This is very similar. Here $V = \frac{4}{3}\pi r^3$. The radius is a length, so to first order in $\alpha\Delta T$, $r = r_0\left(1+\alpha\Delta T\right)$. Therefore

$$V = \frac{4}{3}\pi r_0^3 \left(1+\alpha\Delta T\right)^3 \approx V_0(1+3\alpha\Delta T).$$

So again, by comparison with $V = V_0\left(1+\beta\Delta T\right)$, we find $\beta = 3\alpha$.

Solution 2.6: Now that we have seen that $\beta = 3\alpha$ for 2 completely different shapes, there is an obvious question: is this always true? The answer is that, so long as α is the same in every direction (such materials are called *isotropic*), then this is true. Why? Because any 3D shape, no matter how bizarrely shaped, can always be made out of tiny cubes. If you like, you can think of assembling the entire thing out of legos. We know from Exercise 4 that each of those cubes has $\beta = 3\alpha$. If each of the cubes expands according to $\beta = 3\alpha$, then so will the entire shape.

One should note that the factor of 3 is due to working in three dimensions. In two dimensions, $\beta = 2\alpha$ for isotropic materials.

Solution 2.7: The radii of both the inner and outer circles expand when we heat up the annulus. It is tempting — but *wrong* to think that the inner radius will shrink. Let's see why this is. First consider a very thin ring. When you heat this up, it will expand. In other words, the radius will increase. Second consider two concentric circles, a big one and a little one. When you heat them up, they will both expand and remain concentric. An annulus can be thought of as a collection of infinitely many very thin rings, all concentric and right next to each other. When you heat the whole collection up, each ring will expand, including both the inner and outer edges.

Solution 2.8: There is a lot of extra information in this problem, designed to trick you into thinking its a volume problem. However, the simplest way to do this is just using one dimension. The change in the height of the water is simply

$$L_0\alpha\Delta T = L_0\frac{\beta}{3}\Delta T = 50\,\mathrm{m}\cdot\frac{1}{3}2.10\times10^{-4}\,\mathrm{K}^{-1}\cdot 0.5\,\mathrm{K} = 0.175\,\mathrm{cm}.$$

Solution 2.9: This is a good challenge problem, so I will leave it for you to figure out. It often comes up on homework assignments. As a hint, consider each of the strips of metal as a long thin rod with height, but no width.

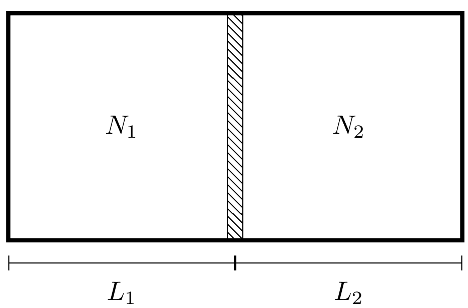
3 The Ideal Gas

3.1 Helpful Equations

$$PV = nRT = Nk_BT. \tag{14}$$

3.2 Problems

Problem 3.1: A box of length 1 m and cross-sectional area A has a moveable partition inside it. There are $N_1 = 3\times10^{23}$ molecules on the left and $N_2 = 2\times10^{23}$ molecules on the right. The gas on both side is in thermal equilbrium at the same temperature T. When the parition settles down to its final position, find the length L_1 and L_2 of the left and right sides of the box.



Problem 3.2: A balloon is filled to volume V at pressure P and temperature V. It is then taken to the bottom of a cold lake (temperature T') and its volume contracts to V'. Assuming that the density of water is a constant ρ, how deep is the lake? *Source: GSI Velan*

Problem 3.3: A helium balloon has volume V_0 and temperature T_0 at sea level where the pressure is P_0 and the air density is ρ_0. The balloon is allowed to float up in the air to altitude z where the temperature is T_1. Show that the volume occupied by the balloon is then

$$V = V_0\left(\frac{T_1}{T_0}\right)e^{cz}$$

where $c = \frac{\rho_0 g}{P_0}$. Assume that the pressure change with altitude is $P = P_0e^{-cz}$. *Source: GSI Sachdeva.*

Problem 3.4: A hollow sphere of radius $R = 0.5\,\mathrm{m}$ contains an ideal gas at temperature $T = 20\,°\mathrm{C}$. By how much should we increase the radius of the sphere if we want to increase the temperature by $10\,°\mathrm{C}$, assuming the pressure stays constant? *Source: GSI Sachdeva.*

4 Kinetic Theory

4.1 Helpful Equations:

$$\langle F(x) \rangle = \int F(x) p(x)\, dx \qquad \text{(Expectation Value of a Function)}$$

$$f(v) = 4\pi N \left(\frac{m}{2\pi k_B T}\right)^{3/2} v^2 \exp\left(\frac{-\frac{1}{2}mv^2}{k_B T}\right) \qquad \text{(Maxwell Distribution)}$$

$$PV = N k_B T \qquad \text{(Ideal Gas Law)}$$

4.2 Problems

Problem 4.1: Suppose that the speed distribution of a collection of N gas molecules is the following "Fake Maxwell Distribution":

$$f(v) = \begin{cases} Av^2 & 0 \le v \le v_0 \\ 0 & \text{otherwise} \end{cases} \tag{15}$$

where A is a positive real constant.

(a) Make a plot of the probability distribution.

(b) Determine the value of A from normalization.

(c) Determine $\overline{v} = \langle v \rangle$.

(d) Show explicitly that $\langle v \rangle^2 \neq \langle v^2 \rangle$. Why is this? What about $\langle v \rangle^3$ and $\langle v^3 \rangle$?

(e) Determine v_p the most probable speed. Does taking a derivative of the distribution help you find this? What is the probability of measuring the speed to actually be v_p?

(f) Determine $v_{\text{rms}} = \sqrt{\langle v^2 \rangle}$.

(g) What is the standard deviation of the speed of these gas molecules?

Problem 4.2: In this problem we will determine the relationship between kinetic energy and temperature for a gas that lives in a 2D world. Consider molecules with mass m that bounce elastically with velocity $\boldsymbol{v}$ inside a square of area L^2.

(a) Consider one wall of the container. What is the average time per molecular collision with that wall? (That is, how many molecules collide with the wall in a one unit of time?)

(b) What is the average change in momentum of a molecule per collusion, $\Delta \boldsymbol{p}$?

(c) What is the average pressure P (force per unit length of wall) on the wall?

(d) If the ideal gas in law in two dimensions is $PA = NkT$, what is the relationship between kinetic energy and temperature in two dimensions?

Problem 4.3: **Note:** This problem is quite challenging.

The energy of a single gas molecule is $K = \frac{1}{2}mv^2$. Show that total energy of a gas governed by the Maxwell distribution is

$$E = \left\langle \frac{1}{2}mv^2 \right\rangle = \frac{3}{2}k_B T.$$

This problem involves a very tricky integral — so tricky, in fact, that a trick is the best way to do it! Try using the identities

$$\frac{d}{d\alpha}\int e^{-\alpha x^2}\, dx = \int \frac{d}{d\alpha} e^{-\alpha x^2}\, dx = \int (-x^2) e^{-\alpha x^2}\, dx \text{ and } \int_{-\infty}^{\infty} e^{-\alpha x^2}\, dx = \sqrt{\frac{\pi}{\alpha}}.$$

Problem 4.4: **Note:** This problem is quite challenging.

At extremely low temperatures, the usual methods of cooling a gas do not work. Instead, a form of evaporative cooling is used, and it is based on the fact that for a Maxwell speed distribution, the fastest $p_N = 10\%$ of the molecules carry $p_E = 28\%$ of the energy of the gas. By removing these atoms and waiting for the gas to come back to equilibrium, you can cool the gas a fixed amount in each evaporation cycle.

(a) The fastest 10% of molecules will correspond to molecules with speeds larger than some value of α. Write down (but do not solve!) the equation that you would use to determine α. *This is simple to write down, but hard to solve so do not try to solve it!*

(b) Write down (but do not evaluate) the expression to determine what percentage of the energy the molecules with speeds larger than α carry. That is, find out how we could compute the 28% mentioned above.

(c) Suppose a gas has initial temperature T_0. What is the temperature of the gas T_1 after one cycle of cooling?

(d) Suppose you wanted to cool the gas to below $\frac{T_0}{2}$. What is the minimum number of cycles needed?

Source: This problem is due to Lenny Evans.

4.3 Solutions

Solution 4.1:

(a) The distribution looks like a quadratic equation going through the origin.

(b) We require the total probability to be 1 (in other words, we have a 100% chance of *something* happening). So

$$1 = \int_0^{v_0} Av^2\, dv = A\frac{v^3}{3}\bigg|_0^{v_0} = \frac{Av_0^3}{3}.$$

Therefore $A = \frac{3}{v_0^3}$.

(c) By definition, the expectation (mean) value of a function $g(v)$ is

$$\overline{g} = \langle g \rangle = \int_0^{v_0} g(v)f(v)\, dv.$$

The bracket and overline are two different notations for the same quantity. In this case, $g(v) = v$, so

$$\overline{v} = \langle v \rangle = \int_0^{v_0} vAv^2\, dv = \frac{Av_0^4}{4} = \frac{3}{v_0^3}\frac{Av_0^4}{4} = \frac{3}{4}v_0.$$

So the average velocity is $\langle v \rangle = \frac{3}{4}v_0$, which makes sense because it has units of velocity.

(d) The point of this question is to see that you cannot bring exponents outside of expectation values (brackets).[1] First we compute the expectation of v^2:

$$\langle v^2 \rangle = \int_0^{v_0} v^2 Av^2\, dv = A\frac{v_0^5}{5} = \frac{3}{v_0^3}\frac{v_0^5}{5} = \frac{3}{5}v_0^2.$$

However,

$$\langle v \rangle^2 = \left(\frac{3}{4}v_0\right)^2 = \frac{9}{16}v_0^2 \neq \frac{3}{5}v_0^2.$$

This is because squaring inside and outside the exponent are fundamentally different things. When we think about $\langle v \rangle$, we are essentially "weighting" the distribution linearly. So the value of $f(2v)$ is twice as important as $f(v)$ in the integral. But for $\langle v^2 \rangle$, we are weighting quadratically, so $f(2v)$ is four times as important as $f(v)$. Even after we square, $\langle v^2 \rangle$ is much more sensitive to what's happening at the high end of the distribution than $\langle v \rangle^2$.

Similar logic holds for $\langle v \rangle^3$ and $\langle v^3 \rangle$.

[1] Expectation is linear, which means that if $g(v)$ and $h(v)$ are two functions, and a, b are real numbers, then $\langle ag + bh \rangle = a\langle g \rangle + b\langle h \rangle$. However, $\langle gh \rangle \neq \langle g \rangle \langle h \rangle$, in most situations.

(e) The most probable speed, by definition, is the global maximum of the probability distribution (the "mode" of the continuous probability distribution). So, by looking at the graph, $v_p = v_0$. In general, the strategy for finding a global maximum is to compute the derivative of the function, identify the zeros of the derivative, and check the values of the function at each of those zeros *and the endpoints of the function* to find the global maximum. In this case, it is pretty clear that the maximum value is at the endpoint just from the graph, so computing the delineative is unnecessary.

(f) From our earlier work, $v_{\text{rms}} = \sqrt{\langle v^2 \rangle} = \sqrt{\frac{3}{5}} v_0$.

(g) By definition, the standard deviation squared (also known as the *variance*) is the average distance between a value and the mean value:

$$\begin{aligned}
\sigma^2 &= \left\langle (v - \langle v \rangle)^2 \right\rangle \\
&= \int (v - \langle v \rangle)^2 f(v)\, dv \\
&= \int \left(v^2 - 2v \langle v \rangle - \langle v \rangle^2 \right) f(v)\, dv \\
&= \int v^2 f(v)\, dv - 2 \langle v \rangle \int v f(v)\, dv + \langle v \rangle^2 \int f(v)\, dv \\
&= \left\langle v^2 \right\rangle - \langle v \rangle^2 .
\end{aligned}$$

So in this case:

$$\sigma = \sqrt{\langle v^2 \rangle - \langle v \rangle^2} = \sqrt{\frac{3}{5} v_0^2 - \frac{9}{16} v_0^2} = \sqrt{\frac{3}{80} v_0^2} = \sqrt{\frac{3}{80}} v_0.$$

Solution 4.2:

(a) Let's think about one molecule at a time, call it molecule "i". If the molecule hits the right hand wall, then from conversation of energy and momentum, it's x-momentum will be reversed and it's y-momentum will be unchanged. We can therefore seperate out the x and y components of the motion. Let the x-velocity of the particle be $v_{x,i}$. Then if the particle hits the right-hand wall it will travel left with velocity $-v_{x_i}$ until it hits the left wall, then bounce back with velocity $v_{x,i}$. Altogether, it must traverse the length of the box twice, a distance $2L$ at a *speed* $v_{x,i}$. Therefore the time between collisions of the particle *with the right-hand wall only* is

$$\Delta T_i = \frac{2L}{v_{x,i}}.$$

(b) The ith particle hits the wall with momentum $mv_{x,i}$ and leave the wall with momentum $-mv_x$, so $\Delta p_i = 2mv_{x,_i}$.

(c) Most of the time, the molecule is exerting no force on the wall, but every round trip it will hit the right-hand wall once and transfer momentum. Since there are so many particles and they are moving very fast, we can think instead about the time-averaged force on the wall from each particle; since *some* particle is always hitting the wall, everything gets averaged out. The time-averaged force on the wall from the ith particle will be

$$F_i = \frac{\Delta p_i}{\Delta T_i} = \frac{2mv_{x,i}}{\frac{2L}{v_{x,i}}} = \frac{mv_{x,i}^2}{L}.$$

The average force on the wall from a *single* molecule is therefore

$$\langle F \rangle = \frac{1}{N} \sum_i F_i = \frac{1}{N} \sum_i \frac{mv_i^2}{L} = \frac{1}{NL} \left(m \langle v_x \rangle^2 \right).$$

This looks almost like the average kinetic energy, but it involves only v_x^2 instead of $v^2 = v_x^2 + v_y^2$. On the other hand, since the square is the same in both directions, the y direction should be exactly like the

x-direction: $\langle v_x^2 \rangle = \langle v_y^2 \rangle$. This implies $\langle v^2 \rangle = \langle v_x^2 + v_y^2 \rangle = 2 \langle v_x^2 \rangle$. Putting this back into the equation for the force,

$$\langle F \rangle = \frac{1}{NL}\left(m\frac{\langle v^2 \rangle}{2}\right) = \frac{1}{NL}\left\langle \frac{1}{2}mv^2 \right\rangle.$$

Knowing this, the pressure is

$$P = \frac{F_{\text{tot}}}{L} = \frac{N\langle F \rangle}{L} = \frac{N}{L^2}\left\langle \frac{1}{2}mv^2 \right\rangle = \frac{1}{L^2}\langle U \rangle,$$

where U is the internal energy of the system, in this case the sum of the kinetic energies of all the molecules.

(d) The ideal gas law says $NkT = PA$, so

$$NkT = PL^2 = \frac{1}{L^2}\langle U \rangle L^2.$$

This gives us the so-called Equipartition Theorem

$$\frac{2}{2}NkT = \langle U \rangle = N\langle \frac{1}{2}mv^2 \rangle = N\frac{1}{2}mv_{\text{rms}}^2.$$

In general, for a ideal gas in a d-dimensional box, the theorem says:

$$\frac{d}{2}NkT = \langle U \rangle = N\frac{1}{2}mv_{\text{rms}}^2.$$

Solution 4.3: If you are riding in the train, then it will appear that the molecules are all moving backwards at velocity $-v$. Observed from rest, the x, y and z distributions of molecules are all the same;

$$f_x(v_x) \propto e^{-\alpha v_x^2} f_y(v_z) \propto e^{-\alpha v_z^2} f_z(v_z) \propto e^{-\alpha v_z^2}$$

for $\alpha = \frac{1}{2}m/kT$. (Multiplying all these together gives the standard Maxwell Distribution.) Said in words, the molecules are moving backwards or forwards in each direction with equal probability. When we are on the moving train, the molecules in the x-direction all *look like* they're moving backwards a speed v fast than before. The average velocity in the x-direction is now $-v$ instead of 0. Because of this bias, the rms speed (no direction) of the molecules is increased. Since the rms speed determines the temperature, the temperature would look higher than it otherwise would.

Solution 4.4: This is a fun integral trick that comes up pretty often in physics. Integrals of the type $\int x^n e^{-\alpha x^2}\,dx$ are called *Gaussian integrals.* Something like half of quantum mechanics can be done using this integral alone.

Let's recall our Maxwell distribution:

$$f(v) = 4\pi\left(\frac{m}{2\pi kT}\right)v^2 e^{-\frac{1}{2}mv^2/kT}.$$

Then from the definition of expectation,

$$\langle U \rangle = \left\langle \frac{1}{2}mv^2 \right\rangle = \frac{1}{2}m\int_0^\infty v^2 4\pi\left(\frac{m}{2\pi kT}\right)v^2 e^{-\frac{1}{2}mv^2/kT}\,dv.$$

This looks worse than it actually is because of all the constants. Define

$$A := \frac{1}{2}m4\pi\left(\frac{m}{2\pi kT}\right)^{3/2} \text{ and } B = \frac{1}{2}m/kT.$$

Then we want to know

$$\langle U \rangle = A\int_0^\infty v^4 e^{-Bv^2}.$$

Using the given identity from 0 to ∞ instead of $-\infty$ to ∞ (this can be proven by going to polar coordinates),

$$\int_0^\infty e^{-\alpha x^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}.$$

Differentiating both sides twice,[2]

$$\frac{d^2}{d\alpha^2}\int_0^\infty e^{-\alpha x^2}\,dx = \int (-x^2)^2 e^{-\alpha x^2}\,dx = \int x^4 e^{-\alpha x^2}\,dx$$

but also

$$\frac{d^2}{d\alpha^2}\frac{1}{2}\sqrt{\frac{\pi}{\alpha}} = \frac{1}{2}\sqrt{\pi}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\alpha^{-5/2} = \frac{3}{8}\sqrt{\frac{\pi}{\alpha^5}}.$$

So

$$\int_0^\infty x^4 e^{-\alpha x^2}\,dx = \frac{3}{8}\sqrt{\frac{\pi}{\alpha^5}}.$$

In our case, $\alpha = B$, so

$$\langle U\rangle = A\frac{3}{8}\sqrt{\frac{\pi}{B^5}}\frac{1}{2}m4\pi\left(\frac{m}{2\pi kT}\right)^{3/2}\sqrt{\pi\left(\frac{2kT}{m}\right)^5} = \frac{3}{2}kT,$$

where the last step is just algebraic simplification.

Solution 4.5: This solution is from Lenny Evans.

(a) The number of particles with speed $v > \alpha$ is

$$N(v>\alpha) = \int_\alpha^\infty f(v)\,dv.$$

If we would like the fastest 10% of particles, we just set this number equal to $N/10$:

$$\frac{N}{10} = \int_\alpha^\infty f(v)\,dv.$$

As stated in the problem, this is a pretty hard equation to solve.

(b) Now we would like to find the energy the fastest 10% of particles carry. In this case, all the energy is kinetic, so we can get is from the Maxwell distribution:

$$U(v>\alpha) = \int_\alpha^\infty \frac{1}{2}mv^2 f(v)\,dv.$$

Thus the fraction of the total energy is

$$p_E = \frac{U(v>\alpha)}{\langle U\rangle} = \frac{\int_\alpha^\infty \frac{1}{2}mv^2 f(v)\,dv}{\int_0^\infty \frac{1}{2}mv^2 f(v)\,dv}.$$

(c) Since the fastest $p_N = 0.1$ fraction of the gas carries $p_E = 0.28$ of the total energy, we know that after cooling

$$U_1 = (1-p_E)U_0.$$

We also know that, from the equipartition theorem,

$$U = \frac{3}{2}NkT,$$

both before and after, and also $N_1 = (1-p_N)N_0$. Therefore, substituting for U_1 and U_0,

$$\frac{3}{2}N_1kT_1 = \frac{3}{2}(1-p_N)N_0kT_1 = (1-p_E)\frac{3}{2}N_0kT_0 \implies (1-p_N)T_1 = (1-p_E)T_0.$$

Solving,

$$T_1 = \frac{1-p_E}{1-p_N}T_0 = \frac{4}{5}T_0.$$

[2]Shockingly, this is completely mathematically rigorous as long as the integral convergences absolutely. This can be proven using Fubini's theorem together with the fundamental theorem of calculus.

(d) If we repeat this process N times, the temperatures is

$$T_n = \left(\frac{4}{5}\right)^n T_0.$$

To be less than half the original temperature, we have the condition that $(0.8)^n \leq \frac{1}{2}$. Therefore

$$n \geq \frac{\log \frac{1}{2}}{\log \frac{4}{5}} \approx 3.1.$$

So n should be at least 4.

5 First Law & Calorimetry

Calorimetry problems are often very realistic in that they involve real materials whose coefficients of specific heat and known and can be found in tables. Many problems therefore ask for numerical answer.

Strategy: when you encounter a problem that expects a numerical answer, solve the problem symbolically and then plug in numerical values in the last step.

In the long run, this ends up being both *causing fewer mistakes* and *quicker*. Why is this? Suppose you had a problem where $c = 0.432\,85\,\mathrm{J\,kg^{-1}\,{}^\circ C^{-1}}$. Then every time you wrote the expression $Q = mc\Delta T = 0.43285 m\Delta T$ you would have write down 6 numerals. Not only does this take much longer than writing c — especially if it must be done in every step of a 10 step calculation — but it is quite likely that you might accidentally write 0.43295 instead of 0.43285 along the way. Additionally, exams usually ask purely symbolic questions, so this strategy is good practice. Therefore it is better to keep everything symbolic until the last step when solving problems.

5.1 Helpful Equations

$$dU = đQ + đW \qquad \text{(First Law of Thermodynamics)}$$
$$đQ = mc\,dT = C\,dT \qquad \text{(Heat Capacity)}$$
$$đQ = L\,dm \qquad \text{(Latent Heat)}$$

5.2 Problems

Problem 5.1: The specific heat of a material varies with temperature as $c(T) = aT$ where $a = 1800\,\mathrm{J\,kg^{-1}\,K^{-2}}$. Calculate the heat required to increase the temperature of 0.250 kg of the material from $T_i = 32\,{}^\circ\mathrm{C}$ to $T_f = 64\,{}^\circ\mathrm{C}$.

Problem 5.2: A cubic iron container with mass $M = 0.5\,\mathrm{kg}$ and length $L = 1\,\mathrm{m}$ is half full of a strange liquid with density $\rho = 300\,\mathrm{kg\,m^{-3}}$. The system is originally at temperature $T = 20\,{}^\circ\mathrm{C}$. We then drop a very hot aluminum pellet of mass $m = 0.1\,\mathrm{kg}$ and temperature $T = 200\,{}^\circ\mathrm{C}$ in the liquid. The specific heats iron, aluminum, and the liquid are $450\,\mathrm{J\,kg^{-1}\,K^{-1}}$, $900\,\mathrm{J\,kg^{-1}\,K^{-1}}$ and $750\,\mathrm{J\,kg^{-1}\,K^{-1}}$ respectively. Calculate the final temperature of the system.

Problem 5.3: At a crime scene, the forensic investigator notes that a 7.2 g lead bullet that was stopped in a door frame apparently melted completely on impact. Assuming the bullet was shot at room temperature (20 °C), what does the investigator calculate as the minimum muzzle velocity of the gun?

Source: Giancoli, 19.25

Problem 5.4: A 2.0 g lead bullet moving at a speed of $200\,\mathrm{m\,s^{-1}}$ becomes embedded in a 2.0 kg wooden block of a ballistic pendulum. Calculate the rise in temperature of the bullet, assuming that all the heat generated raises the bullet's temperature. (Yes, there is enough information to do this problem!)

Source: Halliday and Resnick

Problem 5.5: A copper sample ($c_{\text{Cu}} = 387\,\text{J}\,\text{kg}^{-1}\,\text{K}^{-1}$) of mass $m_c = 75\,\text{g}$ and temperature $T_c = 312\,°\text{C}$ is dropped into a glass beaker that contains a mass of water $m_w = 220\,\text{g}$ ($c_w = 4190\,\text{J}\,\text{kg}^{-1}\,\text{K}^{-1}$). The *heat capacity* (the specific heat times the mass) of the beaker is $C'_b = 190\,\text{J}\,\text{K}^{-1}$. The initial temperature of both the water and the beaker is $T_{w,b} = 12.0\,°\text{C}$. What is the final temperature of the copper, beaker, and water?
Source: 7B Workbook

Problem 5.6: What mass of steam at $100\,°\text{C}$ must be mixed with $150\,\text{g}$ of ice at $-10\,°\text{C}$, in a thermally insulated container, to produce liquid water at $50\,°\text{C}$?

Some useful quantities are $L_v = 2256\,\text{kJ}\,\text{kg}^{-1}$, $L_f = 333\,\text{kJ}\,\text{kg}^{-1}$, $c_w = 4190\,\text{J}\,\text{kg}^{-1}\,\text{K}^{-1}$, and $c_{\text{ice}} = 2220\,\text{J}\,\text{kg}^{-1}\,\text{K}^{-1}$.
Source: 7B Workbook

Problem 5.7: Suppose the specific heat of a material is given by $T = k\frac{T}{T_0}$. How much heat is required to change a mass m of that material from a temperature T_1 to T_2? *Source:* 7B Supplement, pg. 29.

Problem 5.8: **Note:** This problem is quite challenging.

Consider two identical homogeneous balls with the same initial temperature. One of them is at rest on a horizontal plane, while the second hangs from a thread. The same quantities of heat are supplied to both balls. Are the final temperatures of the balls the same or not? If not, calculate the temperature difference.
Source: International Physics Olympiad.

Problem 5.9: **Note:** This problem is easier than it looks.

(a) A rubber ball of mass m and radius r, and specific heat c is rolled off a cliff 10 m above the ground. It bounces back up to a height of 9 m. Assuming all the energy lost in the collision goes into the ball. During the collision, has heat been added to the ball? Has work been done on it? What is the temperature of the ball after the collision?

(b) If the ball always bounces to 90% of the starting height, what is the temperature of the ball after it stops bouncing? (*Hint*: there is way to do this without summing the series.)

(c) Suppose that we now take into account the expansion of the ball due to heating. If then linear expansion coefficient of the ball is α, then what is the final temperature?

Source: GSI Parker

5.3 Solutions

Only symbolic answers will be given.

Solution 5.1: The heat required is

$$Q = \int đQ = \int_{T_i}^{T_f} mc(T)\,dT = \int_{T_i}^{T_f} maT\,dT = \frac{ma}{2}\left(T_f^2 - T_i^2\right).$$

Caution: units matter here! If you use celsius instead of Kelvin you will get the wrong answer! Why? Because $T_f - T_i$ is the same in both celsius and Kelvin, but $T_f^2 - T_i^2$ is not!

Problem 5.10: We use conversation of energy; we list all possible processes of heat transferral and work done, and set them equal to set, then solve for the quantity of interest. We neglect the small amount of work done by dropping the pellet, and use the fact that the container, liquid, and pellet are all the same temperature, T_f, at the end.

$$\begin{aligned}
0 &= \Delta U = \int đQ_{\text{liq}} + \int đQ_{\text{iron}} + \int đQ_{\text{pellet}} \\
&= \int_{T^0_{\text{liq}}}^{T_f} \rho V c_{\text{liq}} dT + \int_{T^0_{\text{Fe}}}^{T_f} M c_{\text{Fe}} dT + \int_{T^0_{\text{Al}}}^{T_f} m c_{\text{Al}} dT \\
&= \rho V c_{\text{liq}}\left(T_f - T^0_{\text{liq}}\right) + M c_{\text{Fe}}\left(T_f - T^0_{\text{Fe}}\right) + m c_{\text{Al}}\left(T_f - T^0_{\text{Al}}\right).
\end{aligned}$$

We can then solve for the final temperature T_f:

$$T_f = \frac{\rho V c_{\text{liq}} T^0_{\text{liq}} + M c_{\text{Fr}} T^0_{\text{Fe}} + m c_{\text{Al}} T^0_{\text{Al}}}{\rho V c_{\text{liq}} + M c_{\text{Fr}} + m c_{\text{Al}}}.$$

Because the pellet is very small and the tank is very large, the final temperature of the tank will not be very different from the starting temperature. When setting up this problem, the first line of the problem is unnecessary after practice, but writing the integrals with bounds is a good way to make sure your signs make sense.

Solution 5.2: When the bullet hit the door frame, its kinetic energy was converted into heat. It must have had at least enough energy to raise its own temperature to its melting point and then melt. It may also have had additional heat which heated the door frame, but there is no way to know how much. Therefore the *minimum* kinetic energy the bullet possessed was the amount needed to melt itself.

From conversation of energy:

$$\begin{aligned} 0 &= Q + W = \int đQ_{\text{heating}} + \int đQ_{\text{melting}} + W_{\text{stopping}} \\ &= \int_{T_{\text{room}}}^{T_{\text{Pb melting}}} m c_{\text{Pb}} dT + \int_0^m L_{\text{Pb melt}} dm - \frac{1}{2} m v^2 \\ &= m c_{\text{Pb}} \left(T_{\text{Pb melting}} - T_{\text{room}} \right) + m L_{\text{Pb melt}} - \frac{1}{2} m v^2. \end{aligned}$$

Therefore

$$v = \sqrt{2 \left(c_{\text{Pb}} \left(T_{\text{Pb melting}} - T_{\text{room}} \right) + L_{\text{Pb melt}} \right)}.$$

Notice that mass entirely cancels out; this is a good reason to do this problem symbolically, because it was never necessary to write down the decimal value of the mass, even finding a numerical answer. Is it reasonable that a bullet had this much energy?

Solution 5.3: This is along the same lines as the last problem. Let the mass of the bullet be m, the mass of the block be M, and the velocity of the block-bullet system be V, immediately after impact. From conversation of energy,

$$0 = \Delta U = Q + W = m c_{\text{Pb}} \Delta T - \frac{1}{2} m v^2 + \frac{1}{2} (m + M) V^2.$$

From conversation of momentum, $mv = (m + M)V$, so

$$\frac{1}{2} (m + M) V^2 = \frac{1}{2} \frac{m^2 v^2}{m + M}.$$

Solving for ΔT yields

$$\Delta T = \frac{1}{m c_{\text{Pb}}} \left(\frac{1}{2} m \left(1 + \frac{m}{m + M} \right) v^2 \right) = \frac{1}{2 c_{\text{Pb}}} \left(1 + \frac{m}{m + M} \right) v^2.$$

Because $M \gg m$, $m/(m + M) \approx 0$, the change in temperature is almost the same as if the large block didn't move at all.

Solution 5.4: This is very similar to exercise 2.

Solution 5.5: This is also very similar to exercise 2, except that we must also take into account the latent heat of vaporization of the steam and the latent heat of melting of the ice.

Solution 5.6: This is very similar to the first question.

Solution 5.7: It turns out that the common solution listed below is subtly wrong and actually violates the second law of thermodynamics. One must take into account the compression of the balls to get the right answer, which involves specific properties of the materials. For a careful explanation, see `https://arxiv.org/pdf/1502.01337.pdf`.

This is a cute brain teaser, but doesn't really involve much physics. The key point is that the center of mass of the ball of the ground will go up as it expands, while the center of ball of the ball hanging from the string will go down. It is therefore "easier" for the ball on the string to increase in temperature because it can convert gravitational potential to heat.

The radii of both balls expand according to $r = r_0\,(1 + \alpha\Delta T)$, so the change in radius is $\Delta r = \alpha\Delta T$.

For the ball on the ground,

$$Q = mc\Delta T_{\text{gr}} + mg\Delta r = mc\Delta T_{\text{gr}} + mg\alpha\Delta T_{\text{gr}}$$

but for the hanging ball

$$Q = mc\Delta T_{\text{hang}} - mg\Delta r = mc\Delta T_{\text{gr}} - mg\alpha\Delta T_{\text{hang}}$$

Since Q is the same for both, one may subtract the two equations to determine $\Delta T_{\text{hang}} - \Delta T_{\text{gr}}$ in terms of the variables defined above.

Solution 5.8:

(a) From conversation of energy, 10% of the potential energy is converted into heat. So

$$0 = \Delta U = mc\Delta T - \frac{1}{10}mg(h-r) \implies \Delta T = \frac{mgh}{10mc} = \frac{gh}{10c}.$$

(b) After it stops bouncing all the gravitational potential has been converted to heat. Following the same logic as the last part, except without the factor of 1/10 gives $\Delta T = \frac{g(h-r)}{c}$.

(c) Conservation of energy gives

$$0 = \Delta U = mc\Delta T - mg(h - r - r\alpha\Delta T),$$

because the center of mass of the ball has risen slightly. One can solve for ΔT and see it will be slightly smaller than in the last part.

6 Heat Conduction

There aren't very many problems on heat conduction around, and it isn't stressed very much. Some years it isn't done at all, but there's often a homework problem on the heat conduction of a cylinder.

6.1 Problems

Problem 6.1: Assume that the thermal conductivity of copper, k, is twice that of aluminium and four times that of brass. Three metal rods, made of copper, aluminium, and brass respectively, are of length ℓ and diameter d. These rods are palced end-to-end with the aluminium between the other two. The free ends of the copper and brass rods are maintained at 100 °C and 0 °C, respectively. Find the equilibrium temperatures of the copper-aluminium and aluminium-brass junctions.

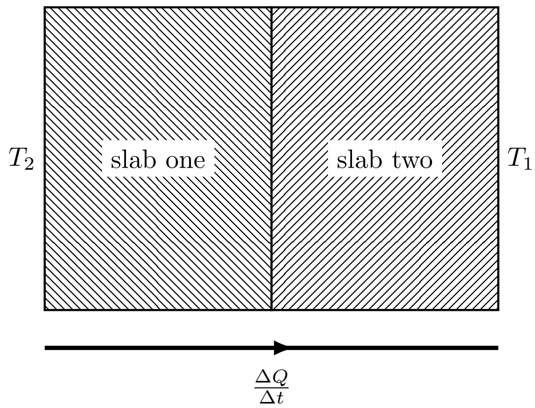
Source: Halliday & Resnick, Ch 22, Problem 17, 1966 Edition.

Problem 6.2: Assuming k is constant, show that the radial flow of heat in a substance between two concentric spheres, of radii $r_1 < r_2$,is given by

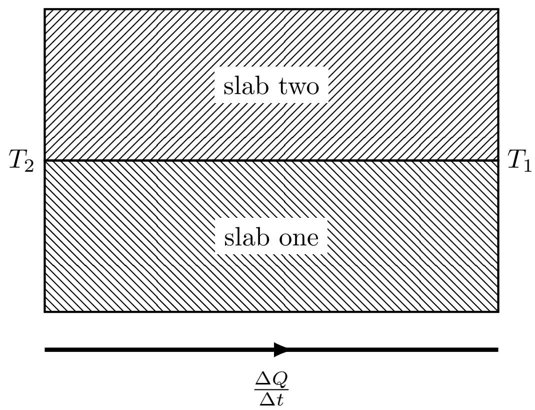
$$\frac{\Delta Q}{\Delta t} = 4\pi k\,(T_1 - T_2)\,\frac{1}{\frac{1}{r_1} - \frac{1}{r_2}}.$$

Source: Halliday & Resnick, Ch 22, Problem 18, 1966 Edition.

Problem 6.3: Suppose there are two slabs with cross-sectional area A and widths L_1 and L_2. Suppose they have thermal conductivities k_1 and k_2 respectively. If the left-hand side of slab one is at T_2, and the right-hand slab two is at T_1, and the right-hand part of slab one is in contact with the left-hand side of slab two, then find the amount of heat $\frac{\Delta Q}{\Delta t}$ that passes through both of the slabs in series.



Problem 6.4: What if the slabs are instead arranged in parallel?



6.2 Solutions

Solution 6.1: The amount of heat conducted per unit time from the copper to the aluminium is the same as from the alumium to the brass. Let this be $\frac{\Delta Q}{\Delta t}$. Let T_h and T_c be the temperatures of the copper and brass ends, and let T_x and T_y be the copper-aluminium and aluminium-brass junction temperatures. Then

$$\frac{\Delta Q}{\Delta t} = \frac{kA\left(T_x - T_h\right)}{\ell} = \frac{(k/2)A\left(T_y - T_x\right)}{\ell} = \frac{(k/4)A\left(T_x - T_c\right)}{\ell}.$$

A judicious application of algebra shows

$$T_x = \frac{1}{7}\left(6T_h + T_c\right) \text{ and } \frac{1}{7}\left(4T_h + 3T_c\right).$$

Solution 6.2: Starting from Fourier's Law,

$$\frac{\Delta Q}{\Delta t} = -kA\frac{dT}{dr}.$$

Note that here $A = A(r)$ and $\frac{\Delta Q}{\Delta t}$, which is the amount of heat conducted by a spherical shell, is not a function of r. We can then integrate this from r_1 to r_2 as

$$\frac{\Delta Q}{\Delta t}\frac{dr}{A(r)} = -kdT \implies \frac{\Delta Q}{\Delta t}\int_{r_1}^{r_2}\frac{dr}{4\pi r^2} = -k\int_{T_1}^{T_2} dT.$$

So

$$\frac{\Delta Q}{\Delta t}\frac{1}{4\pi}\left(\frac{1}{r_2}-\frac{1}{r_1}\right) = -k\left(T_2 - T_1\right).$$

Rearranging gives

$$\frac{\Delta Q}{\Delta t} = 4\pi k\left(T_1 - T_2\right)\frac{1}{\frac{1}{r_1}-\frac{1}{r_2}}.$$

Solution 6.3: Since all the heat that leaves slab one must go into and through slab two,

$$\frac{\Delta Q_1}{\Delta t} = \frac{k_1 A(T_M - T_1)}{L_1} = \frac{k_2 A(T_2 - T_M)}{L_2} = \frac{\Delta Q_2}{\Delta t},$$

where T_M is the temperature in the middle. Solving, for T_M and substituting into either equation gives

$$\frac{\Delta Q}{\Delta t} = -\frac{A(T_2 - T_1)}{\frac{L_1}{k_1}+\frac{L_2}{k_2}}.$$

Solution 6.4: This is easier because the heat conducted by each slab simply adds:

$$\frac{\Delta Q}{\Delta t} = \frac{\Delta Q_1}{\Delta t} + \frac{\Delta Q_2}{\Delta t} = -\left(\frac{k_1}{L_1}+\frac{k_2}{L_2}\right)A\left(T_2 - T_1\right).$$

7 Thermal Processes

The workbook is pretty good for this. There are good discussion questions on pages 19-20 and good problems on pages 21-22. There is an excellent challenge problem on page 23, but it takes a very long time to do, probably 2 hours.

7.1 Problems

Problem 7.1: Use the first law of thermodynamics to show that $C_P = C_V + R$ for an ideal gas.

Problem 7.2: Use the equipartition theorem and the first law to show that $C_V = \frac{d}{2}R$ for a constant volume process.

Problem 7.3: Does adding heat to a system always increase the temperature? Why or why not?

Problem 7.4: The Van der Waals Equation of state is an alternative, and more realistic, model for gases than the ideal gas law. It has two extra constants a and b, which give an equations of state

$$\left(P + \frac{aN^2}{V}\right)(V - Nb) = kT.$$

One can think of a as a measure of the attraction between particles, and b as the volume of each particle. Calculate the work done in an isothermal expansion of a Van der Waals gas from (P_1, V_1) to (P_2, V_2).

Problem 7.5: Consider N particles of an ideal gas undergoing a thermodynamic process from (T_0, V_0) to (T_f, V_f). Along the process, the quantity $\frac{T}{V^2}$ is held constant.

(a) What does the process look like on a PV diagram?

(b) Calculate the work done, the change in internal energy, and the change in heat during this process.

Problem 7.6: There are N molecules of an ideal gas with d degrees of freedom. This gas undergoes an adiabatic process starting at (P_0, V_0).

(a) When the volume of the gas is V_1, what is the temperature of the gas?

(b) Suppose that this temperature is changed by a small amount dT. How does this change the volume? Call this small volume change dV.

(c) The thermal expansion coefficient is the change in volume you get when you change the temperature of the gas a small amount. What is the thermal expansion coefficient of the gas when the volume is V_1?

(d) Why is this coefficient negative?

Problem 7.7: At the bottom of a 1000 m high skyscraper, the outside temperature is $T_{\text{bot}} = 30\,°\text{C}$. The objective is to estimate the outside temperature T_{top} at the top. Consider a thin slab of air (ideal nitrogen with $d = 5$) rising slowly to a height z where the pressure is lower, and assume that this slab expands adiabatically so that it's temperature drops to the temperature of the surrounding air.

(a) How the fractional change in temperature, dT/T, related to dP/P during this process? Hint: start with $PV^\gamma = \text{constant}$.

(b) The atmospheric pressure comes from the weight of the air above it. Express the pressure difference dP in terms of dz, the change in height.

(c) What is the resulting temperature at the top of the building? Find both a symbolic and numerical answer.

Data: $m_{N_2} = 4.65 \times 10^{-26}\,\text{kg}$, $g = 9.8\,\text{m}\,\text{s}^{-2}$. *Source:* adapted from International Physics Olympiad.

7.2 Solutions

Solution 7.1: By definition

$$C_P = \left(\frac{\partial Q}{\partial T}\right)_P . \text{ and } C_V = \left(\frac{\partial Q}{\partial T}\right)_V .$$

Therefore

$$\begin{aligned} C_P &= \left(\frac{\partial}{\partial T}(U - W)\right)_P \\ &= \left(\frac{\partial U}{\partial T}\right)_P - P\left(\frac{\partial V}{\partial T}\right)_P \end{aligned}$$

since we are working with $n = 1$, we have $U = \frac{d}{2}RT$ and $PV = RT$, or $V = RT/P$, so

$$\begin{aligned} &= \frac{\partial}{\partial T}\frac{d}{2}RT + P\left(\frac{\partial}{\partial T}\frac{RT}{P}\right)_P \\ &= \frac{d}{2}R + P\frac{R}{P} \\ &= \frac{d}{2}R + R. \end{aligned}$$

However,

$$C_V = \left(\frac{\partial}{\partial T}(U - W)\right)_V = \left(\frac{\partial U}{\partial T}\right)_V = \frac{d}{2}R,$$

since work on a fluid is zero at constant volume. Therefore $C_P = C_V + R$.

Keep in mind that there are different versions of C_V and C_P. Some are for one mole of a substance and others are for one gram or kilogram. Always check the units to make sure what you're doing makes sense!

Solution 7.2: This was actually shown in the last problem.

Solution 7.3: Adding heat does not always increase the temperature. For instance, at a phase transition, one must supply the latent heat of transformation before the temperature of the material will change. Alternatively, all the heat added could go into increasing the volume of the substance and the temperature might remain constant or even decrease.

Solution 7.4: The work done by an isothermal expansion is given by $W = -\int_{V_1}^{V_2} P(V)\,dV$. Therefore our object is to determine $P(V)$ so that we can do this integral:

$$\left(P + \frac{aN^2}{V}\right)(V - Nb) = kT$$
$$\implies \left(P + \frac{aN^2}{V}\right) = \frac{kT}{V - Nb}$$
$$\implies P(V) = \frac{kT}{V - Nb} - \frac{aN^2}{V}.$$

In the last equation, a, b, N, k are constants and not functions of volume. Furtheremore T is a constant since this is an isothermal processes. Therefore we have found pressure as a function of just the volume and can integrate it:

$$W = -\int P(V)\,dV = -\int_{V_1}^{V_2} \frac{kT}{V - Nb} - \frac{aN^2}{V}\,dV = -kT\ln\frac{V_2 - Nb}{V_1 - Nb} + aN^2\ln\frac{V_2}{V_1}.$$

Solution 7.5:

(a) This is quite similar except we're working with an ideal gas. We know that T/V^2 is a constant for this process. Therefore

$$\frac{T}{V^2} = \frac{T_0}{V_0^2} = \frac{T_f}{V_f^2},$$

for any point (T, V) along the process. To plot this on a PV diagram, we again want to find $P(V)$. Starting from the ideal gas law:

$$P = \frac{NkT}{V} = NkV\frac{T}{V^2} = NkV\frac{T_0}{V_0^2}.$$

However, T_0/V_0^2 is just a constant, so we have $P \propto V$. On a PV diagram, this looks like a straight line through the origin with positive slope.

(b) The work done is

$$W = -\int P(V)dV = -\int_{V_0}^{V_F} NkV\frac{T_0}{V_0^2}\,dV = -NkT\frac{T_0}{V_0^2}\frac{V^2}{2}\bigg|_{V_0}^{V_F} = -\frac{NkTT_0}{2}\left(\frac{V_f^2}{V_0^2} - 1\right).$$

The change in internal energy is just

$$\Delta U = \frac{d}{2}Nk\Delta T = \frac{d}{2}(T_f - T_0).$$

The change in heat follows immediately from the first law: $U = Q + W$.

Solution 7.6: From the adiabatic equation, $P(V)V^\gamma = c = P_0V_0^\gamma = P_1V_1^\gamma$, for some constant $c \in \mathbb{R}$. Therefore

$$\begin{aligned} W &= \int_{V_0}^{V_1} P(V)\, dV \\ &= -\int_{V_0}^{V_1} \frac{c}{V^\gamma}\, dV \\ &= \frac{-c}{1-\gamma}\frac{1}{V^{\gamma-1}}\bigg|_{V_0}^{V_1} \\ &= \frac{1}{\gamma-1}\left[\frac{c}{V_1^{\gamma-1}} - \frac{c}{V_0^{\gamma-1}}\right] \\ &= \frac{1}{\gamma-1}\left[\frac{P_1V_1^\gamma}{V_1^{\gamma-1}} - \frac{P_0V_0^\gamma}{V_0^{\gamma-1}}\right] \\ &= \frac{1}{\gamma-1}\left[P_1V_1 - P_0V_0\right]. \end{aligned}$$

Solution 7.7:

(a) We know $PV^\gamma = P_0V_0^\gamma$, so using the ideal gas law, $P = \frac{NkT}{V}$, so

$$P_0V_0^\gamma = P_1V_1^\gamma = \left(\frac{NkT_1}{V_1}\right)V_1^\gamma \implies T_1 = \frac{P_0V_0^\gamma}{NK}\frac{1}{V_1^{\gamma-1}}.$$

(b) If $PV^\gamma = \text{const} = \left(\frac{NkT}{V}\right)V^\gamma = \left(TV^{\gamma-1}\right)Nk$, then $TV^{\gamma-1} = \text{const}$. So

$$dTV^{\gamma-1} + T(\gamma-1)V^{\gamma-2}dV = d\,(\text{const}) = 0, \tag{16}$$

which we can solve for dV to find

$$dV = \frac{-V^{\gamma-1}dT}{T(\gamma-1)V^{\gamma-2}} = -\frac{1}{\gamma-1}\frac{V}{T}dT. \tag{17}$$

(c) This is simply

$$\frac{dV}{dT} = -\frac{1}{\gamma-1}\frac{V_1}{T_1} \tag{18}$$

(d) Notice the coefficient is

$$-\frac{1}{\gamma-1} = \frac{-1}{\frac{d+2}{d}-1} = -\frac{d}{2} < 0. \tag{19}$$

This is negative because when you add energy, the only way for no heat to enter or leave the gas is, as the temperature goes up, to decrease in volume and increase internal energy.

Solution 7.8: This problem is quite challenging and probably harder than any test question that would appear.

(a) Starting from $PV^\gamma = c$,

$$c = PV^\gamma = P\left(\frac{NkT}{V}\right)^\gamma P^{1-\gamma}T^\gamma Nk \implies P^{\gamma-1}T^\gamma = \text{constant}.$$

Differentiating,

$$(\gamma-1)\,P^{\gamma-2}\,dPT^\gamma + P^{\gamma-1}\,(\gamma)\,T^{\gamma-2}\,dT = 0 = (\gamma-1)\left(P^{\gamma-1}T^\gamma\right)\frac{dP}{P} + \gamma\left(P^{\gamma-1}T^\gamma\right)\frac{dT}{T} \implies \frac{dP}{P} = -\frac{\gamma}{\gamma-1}\frac{dT}{T}.$$

(b) Let's consider a thin slab of air from height z to $z+dz$ and area A, for a total area $V = Adz$. The air above presses down with force

$$\text{area} \times \frac{\text{force}}{\text{area}} = AP(z+dz) \approx AP(z) + A\frac{dP}{dz}dz = AP(z) + A\,dP,$$

while the air below presses up (since pressure in the same in every direction) with pressure $AP(z)$. The air is pulled down by its own mass

$$mg = m_{:=N2}Ng = m_{:=N2}\left(\frac{PV}{kT}\right)g = m_{:=N2}\left(\frac{P(z)Adz}{kT(z)}\right)g. \tag{20}$$

This air (unlike the adiabatically rising air we are considering) is inequilibrium with its surroundings and not moving, so Newton's second law gives

$$0 = \text{force from air above} + \text{weight} - \text{force from air below} = AP(z) + A\,dP + \frac{m_{:=N2}gP(z)A}{kT(z)}\,dz - AP(z).$$

Cancelling the $P(z)$'s and A's gives

$$AdP + \frac{m_{:=N2}gP(z)A}{kT(z)}\,dz = 0 \implies \frac{dP}{P} = -\frac{m_{:=N2}g}{kT(z)}\,dz$$

(c) Combining the results from the first two parts,

$$-\frac{\gamma}{\gamma-1}\frac{dT}{T} = \frac{dP}{P} = -\frac{m_{:=N2}g}{kT(z)}\,dz \implies dT = \frac{\gamma-1}{\gamma}\frac{m_{:=N2}g}{k}\,dz. \tag{21}$$

Integrating this gives

$$T(z) = -\frac{\gamma-1}{\gamma}\frac{m_{:=N2}g}{k}z + C.$$

We know $T_{\text{bot}} = T(z=0) = 30\,°\text{C}$, so $C = 30\,°\text{C}$. Putting in numbers gives

$$T(1000\,\text{m}) = 10.2253\,°\text{C}.$$

BONUS SOLUTION: Derivation of the adiabatic equation using differentials.

For an adiabatic process, no heat is exchanged with the environment, so the First Law of Thermodynamics is

$$dU = đQ + đW = đW = -PdV.$$

Meanwhile, we know from the Equipartition theorem that $U(P,V) = \frac{f}{2}NkT = \frac{d}{2}PV$ where f is the number of degress of freedom of the gas. Using the properties of differentials,

$$dU = \frac{\partial U}{\partial P}\,dP = \frac{\partial U}{\partial V}\,dV = \frac{d}{2}V\,dP + \frac{d}{2}P\,dV. \tag{22}$$

Equating these two, we find

$$\frac{d}{2}V\,dP + \frac{d}{2}P\,dV = -PdV.$$

Grouping together all the dP's and dV's yields

$$-\frac{d}{2}V\,dP = \frac{d}{2}P\,dV + P\,dV = \left(\frac{d}{2}+1\right)P\,dV = \frac{d+2}{2}P\,dV.$$

Putting all the P's one one side and V's on the other and rearranging d's gives

$$-\frac{dP}{P} = \frac{\frac{d+2}{2}}{\frac{d}{2}}\frac{dV}{V} = \frac{d+2}{d}\frac{dV}{V} = \gamma\frac{dV}{V}$$

where in the last step I have defined the adiabatic constant $\gamma = \frac{d+2}{d}$.

Integrating both sides gives

$$\int -\frac{dP}{P} = -\log P + \text{const} = \int \gamma\frac{dV}{V} = \gamma\log P + \text{const}.$$

Since a constant minus a constant is a constant, we can rearrange to find

$$\gamma\log P + \log V = \log\left(PV^\gamma\right) = \text{constant}. \tag{23}$$

Exponentiating both sides and using the fact that $e^{\text{const}} = \text{const}$, for some different constant, gives the adiabatic equation:

$$PV^\gamma.$$

8 Entropy

8.1 Helpful Equations:

$$
\begin{aligned}
U &= \frac{d}{2}NkT && \text{(Equipartition Theorem)} \\
PV &= Nk_BT && \text{(Ideal Gas Law)} \\
dU &= đQ + đW && \text{(First Law of Thermodynamics)} \\
đW &= -PdV && \text{(Work On a Gas)} \\
đQ &= mcdT = CdT && \text{(Heat Capacity)} \\
đQ &= Ldm && \text{(Latent Heat)} \\
\eta &= \frac{\text{“What you get out”}}{\text{“What you put in”}} = \frac{W}{Q_1} && \text{(Engine Efficiency)} \\
dS &= \frac{đQ}{T} && \text{(Entropy for Reversible Processes)} \\
\frac{dQ}{dt} &= -kA\frac{dT}{dx}. && \text{(Heat Conduction)}
\end{aligned}
$$

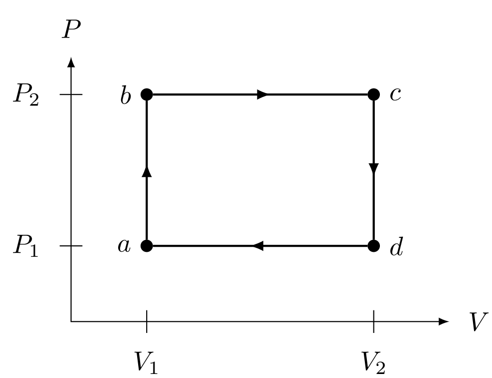
8.2 Problems

Problem 8.1: This problem will develop a useful reference: a list of all the quantities associated with thermodynamic processes of ideal gases.

Suppose that there are N molecules of an ideal gas with d degrees of freedom (use $\gamma = \frac{d+2}{2}$ where it is more convenient). Suppose that the gas starts at (P_0, V_0). Then $T_0 = P_0V_0/(Nk)$. Complete the following table and *draw each process on a P–V diagram.* Skip the last row if you haven't learned about entropy yet. Source: *Workbook, page 153.*

Quantity	Isobaric	Isovolumetric	Isothermal	Adiabatic
P_f		P_F		
V_f	V_F		V_F	
T_f				T_F
ΔU				
Q				
W				
ΔS				

Problem 8.2: Suppose we have a heat engine whose working substance is an ideal gas. Suppose that the engine operators on the following four-step process.



(a) Think about what is going on during step bc. Is heat flowing into the gas our out of the gas during this step? How much? (You only need the first law here)

(b) What about the step cd?

(c) Complete the following table. The entries in the table should be in terms of V_1, V_2, P_1, P_2, and d, the number of degrees of freedom of the gas.

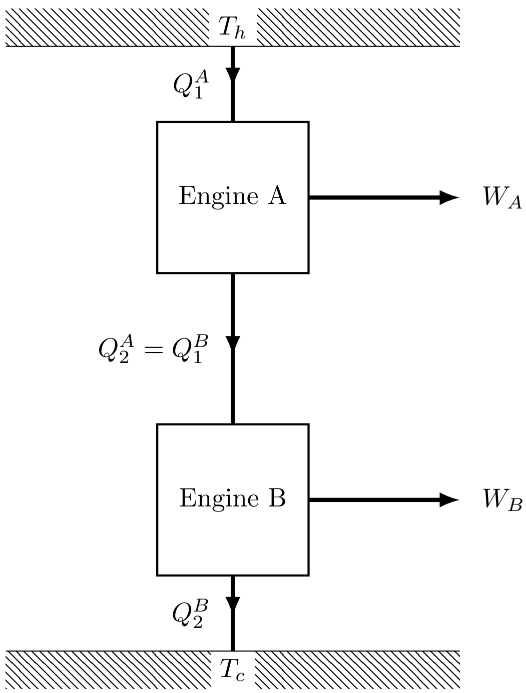
Process	ΔU	ΔQ	ΔW_{on}
ab			
bc			
cd			
da			
Total			

(d) Is the efficiency of this engine given by $\eta = 1 - T_L/T_H$? If so, then why? If not, then calculate the efficiency.

Source: *Workbook, page 27.*

Problem 8.3: In this problem we will see how efficiencies add together. If you have two engines working together, each with efficiency $\eta = \frac{2}{3}$, then the "total engine" clearly cannot have total efficient $\eta_{\text{tot}} = 2\times\frac{2}{3} = \frac{4}{3}$ — that would violate the second law of thermodynamics. This problem will show how we really add the efficiencies.

Consider two engines EA and EB with efficiencies η_A and η_B. We will create a composite engine E by using the output heat of EA to power EB. See the diagram below.



(a) If heat Q_1^A is fed into Engine A, what is the net work output, W_A, and the total heat output Q_2^A from engine A in terms of η_A and Q_1^A?

(b) If the heat input for Engine B is equal to the heat output from Engine A (i.e. $Q_2^A = Q_1^B$), then what is the net work output W_B and the total heat output Q_2^B from Engine B in terms of Q_1^A, η_A and η_B.

(c) What is the total work that is output from both engines?

(d) What is the net efficiency η of the combined system?

(e) Show that if $\eta_A < 1$ and $\eta_B < 1$, then $\eta_{\text{tot}} < 1$.

(f) Suppose Engine A is a Carnot engine operating between temperatures T_H and T_M and Engine B is a Carnot engine operating between temperatures T_M and T_C with $T_H > T_M > T_C$. Show that η, the net efficiency η_{tot} is just the efficiency of a Carnot engine operating between T_H and T_C.

Source: *Modified from the Workbook, page 30.*

Problem 8.4: Suppose N molecules of a monoatomic ideal gas at (P_1, T_1) are held inside a container by a piston with cross-sectional area A. (See Figure below.) The motion of the piston is resisted by a spring with spring constant k. The spring exerts no force in the initial state. The helium is then heated until $T_2 = 3T_1$. Express your answers in terms of the variables given in the problem and fundamental constants. Do not neglect atmospheric pressure — $P_1 \neq 0$!. Determine:

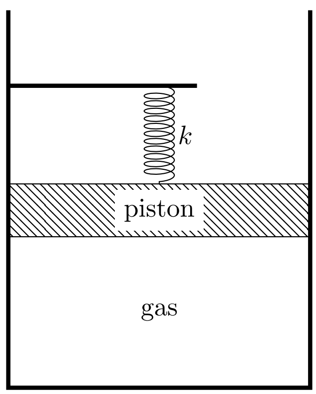
(a) the final volume of the system.

(b) the total work done by the gas.

(c) how much heat was added to the gas.

(d) Some answers are *very messy*. It is a good strategy in cases like this to define your own new variables and use those to keep everything looking nicer. Which *one* new variable should you chose to make your final answers look nice? Write them out in terms of this new variable.

Source: *GSI Lenny Evans.*



8.3 Solutions

Solution 8.1:

Quantity	Isobaric	Isovolumetric	Isothermal	Adiabatic
P_f	P_0	P_f	$\frac{NkT_0}{V_f}$	$P_0\left(\frac{T_0}{T_f}\right)^{\gamma/1-\gamma}$
V_f	V_F	V_0	V_f	$V_0\left(\frac{T_0}{T_f}\right)^{1/\gamma-1}$
T_f	$\frac{P_0V_f}{Nk}$	$\frac{P_fV_0}{Nk}$	T_0	T_f
ΔU	$\frac{d}{2}P_0\Delta V$	$\frac{d}{2}V_0\Delta P$	0	$-\frac{1}{\gamma-1}\left(P_fV_f - P_0V_0\right)$
Q	$\left(\frac{d}{2}+1\right)P_0\Delta V$	$\frac{d}{2}V_0\Delta P$	$NkT_0\ln\frac{V_f}{V_0}$	0
W_{on}	$-P_0\Delta V$	0	$-NkT_0\ln\frac{V_f}{V_0}$	$\frac{1}{\gamma-1}\left(P_fV_f - P_0V_0\right)$
ΔS	$\frac{d+2}{2}Nk\ln\frac{T_f}{T_0}$	$\frac{d}{2}Nk\ln\frac{T_f}{T_0}$	$Nk\ln\frac{V_f}{V_0}$	0

Solution 8.2:

(a) During bc,

$$\Delta W_{\text{on}} = -\int_{V_1}^{V_2} P\,dV = -P_2\left(V_2 - V_1\right) < 0.$$

From equipartition, we know $U = \frac{d}{2}NkT = \frac{d}{2}PV$, so

$$dU = \frac{d}{2}\left(PdV + VdP\right).$$

We can integrate this to find $\Delta U = \frac{d}{2}P_2\left(V_2 - V_1\right)$. Therefore

$$\Delta Q = \Delta U - \Delta W_{\text{on}} = \left(\frac{d}{2} - 1\right) P_2\left(V_2 - V_1\right).$$

(b) This is similar, but now $\Delta W_{\text{on}} = 0$ since V doesn't change, and $\Delta Q = \Delta U = \frac{d}{2}V_2\left(P_2 - P_1\right)$.

(c) The table should look as follows.

Process	ΔU	ΔQ	ΔW_{on}
ab	$\frac{d}{2}V_1\left(P_2 - P_1\right)$	$\frac{d}{2}V_1\left(P_2 - P_1\right)$	0
bc	$\frac{d}{2}P_2\left(V_2 - V_1\right)$	$\left(\frac{d}{2} + 1\right)P_2\left(V_2 - V_1\right)$	$-P_2\left(V_2 - V_1\right)$
cd	$\frac{d}{2}V_2\left(P_1 - P_2\right)$	$\frac{d}{2}V_2\left(P_1 - P_2\right)$	0
da	$\frac{d}{2}P_1\left(V_1 - V_2\right)$	$\left(\frac{d}{2} + 1\right)P_1\left(V_1 - V_2\right)$	$-P_1\left(V_2 - V_1\right)$
Total	0	Not Useful	$-\Delta P \Delta V$

(d) This isn't a Carnot cycle, so the efficiency will not be $\eta = 1 - T_L/T_H$. For one, only a Carnot cycle can achieve that efficiency. For another, there are no well-defined hot and cold temperatures. The efficiency is given by

$$\eta = \frac{W}{Q_{\text{in}}} = 1 - \frac{Q_{\text{out}}}{Q_{\text{in}}} = 1 - \frac{-Q_{\text{cd}} - Q_{\text{da}}}{Q_{ab} + Q_{bc}}.$$

It doesn't simplify to anything particularly nice.

Solution 8.3:

(a) We know $\eta_A = \frac{W_A}{Q_1^A}$ by definition, so $W_A = \eta_A Q_1^A$. By conversation of energy, $Q_1^A = W_A + Q_2^A$, so

$$Q_2^A = Q_1^A - W_A = Q_1^A\left(1 - \eta_A\right).$$

(b) By definition, $\eta_B = \frac{W_B}{Q_1^B}$, so

$$W_B = \eta_B Q_1^B = \eta_B Q_2^A = \eta_B\left(1 - \eta_A\right)Q_1^A.$$

Similarly,

$$Q_2^B = Q_1^B - W_B = \left(1 - \eta_B\right)Q_1^B = \left(1 - \eta_B\right)\left(1 - \eta_A\right)Q_1^A.$$

(c) The total work output is

$$W = W_A + W_B = \eta_A Q_1^A + \eta_B\left(1 - \eta_A\right)Q_1^A = \left(\eta_A + \eta_B(1 - \eta_A)\right)Q_1^A.$$

(d) The efficiency of the whole thing is

$$\eta_{\text{tot}} = \frac{\text{What you get out}}{\text{What you put in}} = \frac{W}{Q_1^A} = \frac{Q_2^B - Q_1^A}{Q_1^A} = \frac{\left(1 - \eta_B\right)\left(1 - \eta_A\right)Q_1^A - Q_1^A}{Q_1^A} = 1 - \left(1 - \eta_A\right)\left(1 - \eta_B\right).$$

(e) Since $\eta_A < 1$ and $\eta_B < 1$, $0 < (1 - \eta_A) < 1$ and $0 \leq \left(1 - \eta_B\right) \leq 1$. Therefore $0 < \left(1 - \eta_A\right)\left(1 - \eta_B\right) \leq 1$, which implies

$$\eta = 1 - \left(1 - \eta_A\right)\left(1 - \eta_B\right) < 1.$$

(f) In this case, $\eta_A = 1 - \frac{T_M}{T_H}$ and $\eta_B = 1 - \frac{T_C}{T_M}$. Therefore

$$\eta = 1 - \left(1 - \left(1 - T_M/T_H\right)\right)\left(1 - \left(1 - T_C/T_M\right)\right) = 1 - \frac{T_M T_C}{T_M T_H} = 1 - \frac{T_C}{T_H},$$

which is the efficiency of a Carnot cycle between T_C and T_H.

Solution 8.4: I've decided this isn't a good problem — it's just too ugly and you don't learn anything. The only "physics" part is to notice that the final pressure P_2 obeys

$$P_{\text{atm}} = P_2 - k\Delta x = P_2 - k\frac{V_1 - V_2}{k}.$$

In principle, one can use $P_{\text{atm}} = P_1$, $P_1 V_1 = NkT_1$, $P_2 V_2 = NkT_2$, and $T_2 = 3T_1$ to find a quadratic equation for V_2 and eliminate all other variables.

9 Second Law & Entropy

Pages 34 and 35 of the workbook are quite good for this. A modified version with updated diagrams is below.

9.1 Helpful Equations:

$$U = \frac{d}{2}NkT \qquad \text{(Equipartition Theorem)}$$
$$PV = Nk_BT \qquad \text{(Ideal Gas Law)}$$
$$dU = đQ + đW \qquad \text{(First Law of Thermodynamics)}$$
$$đQ = mcdT = CdT \qquad \text{(Heat Capacity)}$$
$$đQ = Ldm \qquad \text{(Latent Heat)}$$
$$\eta \leq \frac{\text{"What you get out"}}{\text{"What you put in"}} = \frac{W}{Q_1} \qquad \text{(Engine Efficiency)}$$
$$dS = \frac{đQ}{T}. \qquad \text{(Entropy (for reversible processes))}$$

9.2 Problems

Problem 9.1: Two moles of monoatomic ideal gas expands from point $A = (P, 8V)$ to $B = (P, 32V)$ at constant pressure and in a reversible fashion.

(a) Draw the transformation on a PV diagram. (It is a very good idea to always do this for any problem. It's quick and helps you think visually about what's happening.)

(b) Is heat flowing into the gas or out of the gas during this transformation? That is, is $Q \geq 0$ or $Q \leq 0$?

(c) What is the change of entropy $\Delta S_{A\to B}$ during this transformation? Is the sign consistent with the answer to the last part?

(d) Let $X = (32P, V)$. Now suppose that we have a different process $A \to X \to B$ so that $A \to X$ is adiabatic and $X \to B$ is isothermal. Draw this process on a PV diagram. Compute

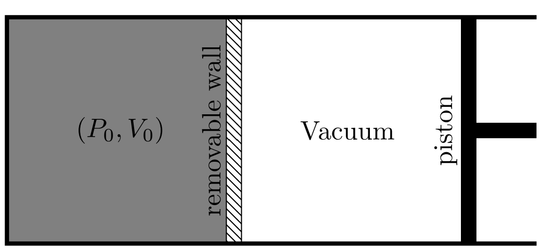
$$\Delta S_{A\to X} = \int_A^X dS = \int_A^X \frac{đQ}{T}.$$

(e) Now find $\Delta S_{X\to B}$ and $\Delta S_{A\to X\to B} = \Delta S_{A\to X} + \Delta S_{X\to B}$.

(f) How does this compare to $\Delta S_{A\to B}$. Why is this?

Source: Workbook, pg 34.

Problem 9.2: Consider the following thermal device.



It goes through the following process:

Step 0 The gas starts at $A = (P_0, V_0)$.

Step 1 The removable wall is removed and the gas expands freely to fill the vacuum. At the end of this process, the gas is in equilibrium at $B = (P_1, 2V_0)$.

Step 2 The piston compresses the gas back to volume V_0. Simultaneously, heat is drawn out of the gas so that this process happens at constant pressure. This process ends at $C = (P_1, V_0)$.

Step 3 The volume of the gas is fixed at V_0 and heat is added back in until the pressure returns to P_0. The removable wall is reinserted. The gas is now at A again and the cycle can be repeated.

(a) Is step 1 reversible? Step 2? Step 3?

(b) Draw as much of a PV diagram as you can for this cycle. What difficultly do you encounter?

(c) Explain why the temperature of the gas at B is the same as the temperature at A.

(d) Using this fact, find P_1.

(e) Why is this device *not* an engine? (Try using a fact about W_{tot}.)

(f) How much entropy is added during each step?

(g) What is the total change in entropy?

(h) How much entropy is added to the *environment* during each step?

(i) What is $\Delta S_{\text{universe}} = \Delta S_{\text{gas}} + \Delta S_{\text{environment}}$ for the whole cycle?

Source: Workbook, pg 34.

Problem 9.3: Suppose you have a mass M of steam at 100 °C, but no heat source to maintain it in that condition. You also have a cold reservoir whose temperature cannot be changed from 0 °C.

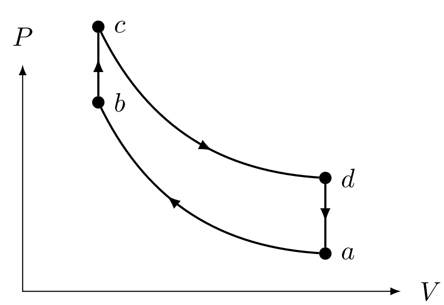
Suppose that you operator a reversible heat engine with this system: the steam condenses and cools until it reached 0 °C, and all of the heat released is used to run the engine. Suppose that the latent heat of vaporization is L_v and the specific heat of water per unit mass is c_w.

(a) Calculate the total entropy change of the steam as it condenses to water and cools to 0 °C.

(b) Find the total amount of work that the engine can do. Carry out any calculations and explain your reasoning. (*Hint*: How much heat must the engine expel to the low temperature reservoir?)

Source: Workbook, pg 34.

Problem 9.4: The operation of an automobile internal combustion engine can be approximated by a reversible cycle known as an Otto cycle, which involved two adiabatic paths and two isovolumetric paths. Assume we are using n moles of an ideal *diatomic* gas as the working substance and assume the system is not hot enough to have appreciable vibrational kinetic energy.



(a) Express the work done by the gas along each branch of the cycle in terms of pressures and volumes.

(b) What is the amount of heat flowing into the gas associated with each of the four processes in this cycle in terms of pressures and volumes.?

(c) What is the change in entropy of the gas along each branch?

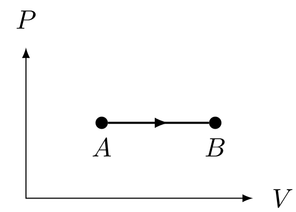
(d) Compute the efficiency of the cycle in terms of V_a and V_b only.

Source: Bordel, Midterm 1, Fall 2012.

9.3 Solutions

Solution 9.1:

(a) The phase diagram looks as follows.



(b) The equipartition theorem says that for a monoatomic gas, $U = \frac{3}{2}NkT = \frac{3}{2}PV$. Therefore

$$dU = \frac{3}{2}d(PV) = \frac{3}{2}(PdV + VdP) \text{ or} \Delta U = \frac{3}{2}[P\Delta V + V\Delta P].$$

So

$$Q = \Delta U - W = \frac{3}{2}[P\Delta V + V\Delta P] + P\Delta V = \frac{5}{2}P\Delta V > 0.$$

The last equality follows from $\Delta P = 0$.

(c) There are several equivalent ways to do this. To start with, we use the fact that this is a reversible process: $dS = \frac{đQ}{T}$. We now want to get rid of $đQ$ and replace it with something we can actually integrate. Since we are working at constant pressure, we can use $đQ = C_P dT = \frac{5}{2}NkdT$. Therefore

$$\Delta S = \int_{T_A}^{T_B} \frac{5}{2}NkdT = \frac{5}{2}Nk\ln\frac{T_B}{T_A} = \frac{5}{2}Nk\ln\frac{P32V}{Nk}\frac{Nk}{P8V} = \frac{5}{2}Nk\ln 4 = 5Nk\ln 2.$$

Alternatively, we can use

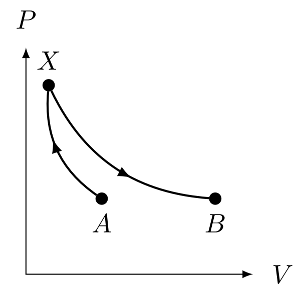
$$đQ = dU - đW = \frac{3}{2}(PdV + VdP) + PdV = \frac{5}{2}PdV + \frac{3}{2}VdP.$$

So the change in entropy becomes

$$\Delta S = \int\frac{đQ}{T}\int_{8V}^{32V}\frac{5}{2}\frac{P}{T}\,dV + \int_P^P \frac{3}{2}\frac{V}{T}\,dP = \frac{5}{2}\int_{8V}^{32V}\frac{Nk}{V}\,dV + 0 = \frac{5}{2}Nk\ln\frac{32V}{8V} = 5Nk\ln 2.$$

Notice that the sign is the same as in the last part.

(d) This process looks something like this.



Since $A \to X$ is adiabatic, $đQ = 0$, so $\Delta S_{A\to X} = 0$.

(e) Let's compute this explicitly for more practice:

$$\begin{aligned}
\Delta S_{X\to B} &= \int \frac{đQ}{T} \\
&= \int \frac{dU - đW}{T} \\
&= \int \frac{3}{2}\frac{Nk}{T}\,dT + \int \frac{P}{T}\,dV \\
&= 0 + \int_V^{32V} \frac{Nk}{V}\,dV \\
&= Nk \ln \frac{32V}{V} \\
&= 5Nk\ln 2.
\end{aligned}$$

So

$$\Delta S_{A\to X\to B} = \Delta S_{A\to X} + \Delta S_{X\to B} = 0 + 5Nk\ln 2.$$

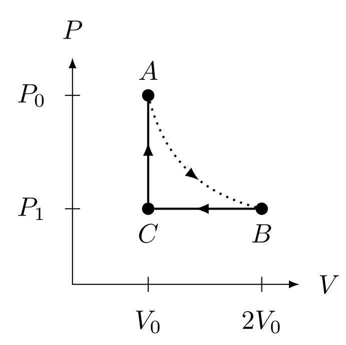
(f) Of course, this is the same as $\Delta S_{A\to B}$, because entropy is a state variable. This gives us a very useful computational trick: if it looks like computing the entropy for some thermodynamic process is going to be a pain, then you can always use any other process or combination of processes whose endpoints are the same. For instance, you can always decompose and change $(P, V) \to (P', V')$ into an isobaric process followed by an isovolumetric process: $(P, V) \to (P, V') \to (P', V')$. It might be easier to compute the entropy of both of these processes separately, depending on what information is given in the problem.

Solution 9.2:

(a) Step 1 is not reversible. One reason for this is the processes is not in equilibrium. Consider a time shortly after the wall is removed. Then most of the gas is still on the left-hand side and the right-hand side is mostly vacuum. So then the pressure on the left-hand wall is slightly less than P_0 and the pressure of the right-hand wall is slightly more than 0, and it varies in between. This means there is not a single, well-defined "pressure" for the gas as a whole, a clear sign that the gas is not in equilibrium.

Steps 2 and 3 are reversible.

(b) The PV diagram looks as follows. Because Step 1 from A to B is not an equilibrium process, and only its endpoints are well-defined equilibrium states, we cannot draw it on the PV diagram. I have denoted it with a dotted line to indicate this.



(c) No heat has been absorbed and no work has been done, so the internal energy does not change. The equipartition theorem implies that the temperature also does not change.

(d) Even though $A \to B$ is not a equilibrium process, the point B is the same *as if* we had taken an isothermal process from V_0 to $2V_0$. Therefore

$$P_0V_0 = P_1V_1 = P_1 2V_0 \implies P_1 = \frac{P_0V_0}{2V_0} = \frac{1}{2}P_0.$$

(e) Let's think about the work done by each step. During step 1, the piston does not move (even though the volume of the gas changes), so no work is done. During step 2, the piston is pushed from $2V_0$ to V_0, so the gas does work $-P_1V_1 < 0$. During step 3, there is no change in the position of the piston, so again no work is done. So overall the gas has done negative work. Since a heat engine in a machine that uses heat to do work, and we have not done (positive) work, this is not a heat engine.

(f) Recall that entropy is a state variable. This means that we know that the change in entropy (of the gas) is zero for the full cycle. It's easiest to calculate the change in entropy for steps 2 and 3 and use that to infer the change in entropy during step one. So:

$$\Delta S_{BC} = \int_B^C \frac{đQ}{T} = \int_B^C \frac{C_P dT}{T} = \frac{d+2}{2} Nk \int_{T_B}^{T_C} \frac{dT}{T} = \frac{d+2}{2} Nk \ln \frac{T_C}{T_B} = \frac{d+2}{2} Nk \ln \frac{P_1 V_0}{P_1 2V_0} = -\frac{d+2}{2} Nk \ln 2.$$

Similarly,

$$\Delta S_{CA} = \int_C^A \frac{đQ}{T} = \int C_V \frac{dT}{T} = \frac{d}{2} Nk \ln \frac{T_A}{T_C} = \frac{d}{2} Nk \ln \frac{P_0V_0}{\frac{1}{2}P_0V_0} = \frac{d}{2} Nk \ln 2.$$

Therefore

$$0 = \Delta S = \Delta S_{AB} + \Delta S_{BC} + \Delta S_{CA} = \Delta S_{AB} + \left(-\frac{d+2}{2} + \frac{d}{2}\right) Nk \ln 2$$

so

$$\Delta S_{AB} = -\left(-\frac{d+2}{2} + \frac{d}{2}\right) Nk \ln 2 = Nk \ln 2.$$

Another way to calculate ΔS_{AB} would be to use the fact $\Delta S_{AB} = S_B - S_A$ depends only on the endpoints. We can draw an isothermal process from A to B and the change in entropy for that process will be the same as the free expansion. [3] Then, using the fact that $đQ = dU - đW = 0 + PdV$, since $dU = 0$ for isothermal processes,

$$\Delta S_{AB} = \Delta S_{AB,\text{isotherm}} = \int \frac{đQ}{T} = \int \frac{PdV}{T} = \int \frac{Nk}{V} dV = Nk \ln \frac{V_B}{V_A} = Nk \ln \frac{2V_0}{V_0} = Nk \ln 2.$$

Of course, this agrees with our other answer.

(g) As mentioned before, $\Delta S_{ABCA} = 0$ because entropy is a state variable.

(h) This is a bit tricky, and I realized that not quite enough information was given in the problem to solve this. Let us make the additional assumption that the system is at the same temperature as its environment. That is to say, the system and its environment are in thermal equilibrium: $T_{\text{gas}} = T_{\text{environment}}$. Of course, this does not make sense during $A \to B$, but we will see this isn't a problem.

During step 1, the environment is isolated from the system and there is no heat transfer. So $\Delta S_{AB,\text{env}} = 0$. During steps 2 and 3, all of the heat Q absorbed by the system comes from the environment: $đQ_{\text{gas}} = -đQ_{\text{environment}}$. Therefore

$$\Delta S_{BC,\text{env}} = -\Delta S_{BC,\text{gas}} \text{ and } \Delta S_{CA,\text{env}} = -\Delta S_{CA,\text{gas}}.$$

Therefore $\Delta S_{\text{env,tot}} = -\Delta S_{BC,\text{gas}} - \Delta S_{CA,\text{gas}} = Nk \ln 2$.

(i) This just adds up the previous parts.

$$\Delta S_{\text{universe}} = \Delta S_{\text{gas}} + \Delta S_{\text{env}} = 0 + Nk \ln 2 = Nk \ln 2.$$

Solution 9.3:

(a) We break the process into two parts: condensing the steam, and cooling the water. For the first, we have $đQ = L_v dm$ and for the second we have $đQ = Mc_w dT$. Then

$$\Delta S = \Delta S_{\text{condense}} + \Delta S_{\text{cool}} = \int \frac{L_v dM}{T} + \int \frac{Mc_w dT}{T} = \frac{L_v}{T_{\text{melt}}} M + Mc_w \ln \frac{T_{\text{melt}}}{T_{\text{freeze}}}.$$

(b) To find the work done, we need the efficiency and Q_{in}. However, all we know is Q_{in}. The best we can do is to use the bound

$$\eta = 1 - \frac{Q_{\text{in}}}{Q_{\text{out}}} \leq 1 - \frac{T_h}{T_c} = 1 - \frac{373}{273} \approx \frac{2}{3}.$$

So the work done is $W = \eta Q_{\in} \leq \left(1 - \frac{T_h}{T_c}\right) Q_{in}$.

[3] This clever trick of replacing a process which is difficult or impossible to calculate with one that can be computed with easily works for any state variable: P, V, T, S, U. It does *NOT* work for Q or W.

Solution 9.4: See `https://tbp.berkeley.edu/exams/3623/download/`, question 5.

10 Continuous Charge Distributions

The general strategy to solve electrostatics problems is as follows:

1. Choose a coordinate system.
2. Find dq in terms of geometric differentials (i.e. $dq = \lambda\, dx$).
3. Choose an arbitrary small chunk of charge dq and find the differential electric field $d\boldsymbol{E}$ due to that chunk of charge. The tricky step here is using the geometry to determine $\boldsymbol{\mathscr{r}}$, the vector from dq to the point you're trying to compute the electric field at.
4. Use superposition to write $\boldsymbol{E} = \int d\boldsymbol{E}$ and integrate.

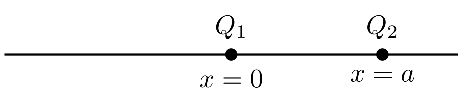
10.1 Helpful Equations

$$\boldsymbol{F} = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{r^2}\hat{r} \qquad \text{(Coulomb's Law)}$$

$$\boldsymbol{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}. \qquad \text{(Electric Field from a Point Charge)}$$

10.2 Problems

Problem 10.1: Suppose there are two point charges $Q_1 = 2q$ and $Q_2 = -3q$ places along a line as shown.



We want to find a place to place another charge Q where it will remain at rest.

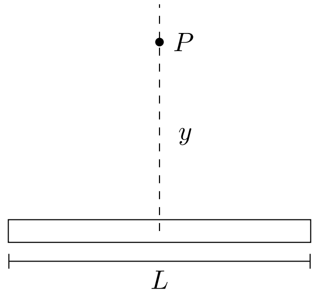
(a) Compute the force on a charge q at a point $0 < x < a$. Remember: force is a vector quantity! It has *both magnitude and direction!* Is there any point between the two charges where the charge q will remain at rest? What if it were a charge $-q$?

(b) Answer the same question for $x > a$.

(c) Ditto for $x < 0$.

Problem 10.2: A thin nonconducting rod of finite length L has a charge q spread uniformly along it. Show that the electric field $\boldsymbol{E}$ at a point P on the line that is perpendicular to the rod and passes through its center is given by

$$\boldsymbol{E}(y) = \frac{q}{2\pi\varepsilon_0}\frac{1}{y\sqrt{L^2 + 4y^2}}\hat{y}.$$



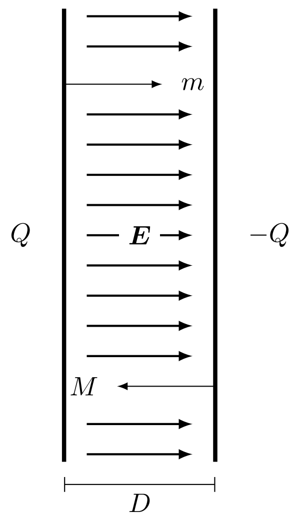
Problem 10.3: Suppose there is a disk of radius R with uniform charge density σ on its surface. Consider a point P on the axis of the disk at a distance z from the center. If the axis of the disk coincides with the z-axis, prove that the electric field is

$$\boldsymbol{E} = \frac{\sigma}{2\varepsilon_0}\left(1 - \frac{z}{\sqrt{z^2 + r^2}}\right)\hat{z}.$$

Problem 10.4: Two large parallel copper plates are a distance D apart and have a uniform electric field $\boldsymbol{E}$ between them (see figure). An electron with mass m is released from the negative plate at the same time that a proton with mass M is released from the positive plate. Neglect the force of the particles on each other and prove that their distance from the positive plate when they mass each other is given by

$$x_{\text{meet}} = D\frac{m}{m + M}.$$

Why does this result not depend on the strength of the electric field?



Problem 10.5: **Note:** This problem is quite challenging.

Two thin rigid rods lie on the x-axis, as shown below. Both rods are uniformly charged. The rods have lengths L_1 and L_2 and charge per unit length λ_1 and λ_2 respectively. The distance between the rods is L.

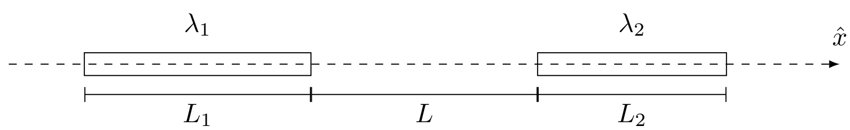
(a) Show that the force on rod 2 exerted by rod 1 is given by

$$\boldsymbol{F}_{1,2} = k\lambda_1\lambda_2 \ln\left[\frac{(L_2 + L)(L_1 + L)}{L(L + L_1 + L_2)}\right]\hat{x} \tag{24}$$

(b) Show that when $L \gg L_1$ and $L \gg L_2$, Equation (24) can be written in the form

$$\boldsymbol{F}_{1,2} = k\frac{Q_1 Q_2}{L^2}\hat{x}.$$

What are Q_1 and Q_2?



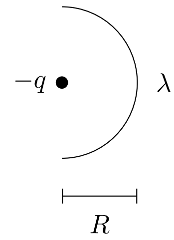
Problem 10.6: A rod with a uniform linear charge density λ is bent int oa half-circle of radius R. A point charge $-q$ is placed at the center of the circle. (The Rod and the point charge are each held fixed in place.)

(a) What is the net charge on the half-circle?

(b) Set up an integral to find the force on the point charge due to the half-circle. Remember that force is a vector!

(c) What direction does the force point? How can you tell without doing any calculations?

(d) Evaluate the integral and find the vector force on the point charge.



Problem 10.7: Consider a cylinder of constant volume charge density ρ. The cylinder has height h and radius R. It is centered at the origin, and its axis lies along the positive z-axis. What is the $\boldsymbol{E}$-field at any point on the positive z-axis with $z > h/2$?

Source: 7B Workbook, page 47.

Problem 10.8: **Note:** This problem is quite challenging.

Calculate the electric field at an axial point z of a thin, hollow uniformly charged cylinder of charge density σ, radius R, and length L. Here, z is the distance measured from the center of the cylinder. (Hint: recall the expression for electric field for a ring of charge at a point on its axis) Answer:

$$\boldsymbol{E} = \frac{\sigma R}{2\varepsilon_0}\left[\frac{1}{\left((z+L/2)^2+R^2\right)^{1/2}} - \frac{1}{\left((z-L/2)^2+R^2\right)^{1/2}}\right]\hat{z}.$$

10.3 Solutions

The general strategy to solve electrostatics problems is as follows:

(a) Choose a coordinate system.

(b) Find dq in terms of geometric differentials (i.e. $dq = \lambda dx$).

(c) Choose an arbitrary small chunk of charge dq and find the differential electric field $d\boldsymbol{E}$ due to that chuck of charge. The tricky step here is using the geometry to determine $\boldsymbol{\mathscr{r}}$, the vector from dq to the point you're trying to compute the electric field at.

(d) Use superposition to write $\boldsymbol{E} = \int d\boldsymbol{E}$ and integrate.

Solution 10.1:

(a) The electric field will be

$$\boldsymbol{E}(0 < x < a) = \frac{kqQ_1}{x^2}\hat{x} - \frac{kqQ_2}{(a-x)^2}\hat{x}.$$

This will be zero where

$$\frac{kqQ_1}{x^2} - \frac{kqQ_2}{(x-a)^2} = 0 \implies \frac{Q_1}{x^2} = \frac{Q_2}{(a-x)^2} \implies \left(1 - \frac{Q_2}{Q_1}\right)x^2 - 2ax + a^2 = 0.$$

If $-q$ were negative, then there would be an overall minus sign on both terms, but the stable position where the charge would stay at rest would not change.

(b) The only difference is that the electric field from Q_2 will now point in the opposite direction, yielding

$$\boldsymbol{E}(x > a) = \frac{kqQ_1}{x^2}\hat{x} + \frac{kqQ_2}{(a-x)^2}\hat{x}.$$

This is zero if there are any solutions to

$$\left(1 + \frac{Q_2}{Q_1}\right)x^2 - 2ax + a^2 = 0.$$

(c) This is the same as part (a), except that there would be an overal minus sign on the first term, so

$$\boldsymbol{E}(x<0) = -\frac{kqQ_1}{x^2}\hat{x} - \frac{kqQ_2}{(a-x)^2}\hat{x}.$$

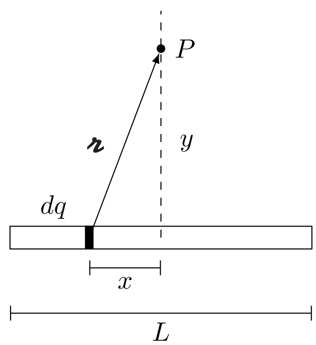
Solution 10.2: We assume that the rod has negligible radius; approximate the rod by a line of charge. We can now apply the four steps above to find the electric field $\boldsymbol{E}(y)$.

(a) Let's choose coordinates where the origin is at the center of the rod, the x-axis is along the rod, and the y-axis is vertical.

(b) We will integrate along the x-axis from $x = -L/2$ to $x = L/2$. The charge dq is then

$$dq = \frac{\text{charge}}{\text{length}} = \frac{Q}{L}dx.$$

(c) Consider a chunk of charge dq at an arbitrary position x. This is shown in the picture below for the case $x < 0$.



From the geometry, we can immediate write $\boldsymbol{\mathscr{r}} = x\hat{x} + y\hat{y}$ (with $x < 0$ as drawn). Then the length of the vector is $\mathscr{r} = |\boldsymbol{\mathscr{r}}| = \sqrt{x^2+y^2}$ by the Pythagorean theorem.

$$d\boldsymbol{E}(x) = \frac{k\,dq}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}} = \frac{k\,dq}{\mathscr{r}^2}\frac{\boldsymbol{\mathscr{r}}}{\mathscr{r}} = \frac{k\,dq}{\mathscr{r}^3}\boldsymbol{\mathscr{r}} = \frac{k\,dq}{(x^2+y^2)^{3/2}}(x\hat{x}+y\hat{y}) = \frac{k\frac{Q}{L}}{(x^2+y^2)^{3/2}}(x\hat{x}+y\hat{y})$$

(d) We can now integrate this to find the electric field:

$$\begin{aligned}
\boldsymbol{E} &= \int d\boldsymbol{E} \\
&= \int_{-L/2}^{L/2} \frac{k\frac{Q}{L}}{(x^2+y^2)^{3/2}}(x\hat{x}+y\hat{y}) \\
&= \frac{kQ}{L}\int_{-L/2}^{L/2}\frac{x}{(x^2+y^2)^{3/2}}\hat{x} + \frac{kQ}{L}\int_{-L/2}^{L/2}\frac{y}{(x^2+y^2)^{3/2}}\hat{y}.
\end{aligned}$$

the first integral is an easy u-substitution, and the second can be done with the trig substitution $x = y\tan\theta$, which gives

$$\begin{aligned}
&= \frac{kQ}{L}\left.\frac{-1}{\sqrt{x^2+y^2}}\right|_{-L/2}^{L/2}\hat{x} + \frac{kQ}{L}\left.\frac{x}{y\sqrt{x^2+y^2}}\right|_{-L/2}^{L/2}\hat{y} \\
&= \frac{kQ}{L}\left(\frac{-1}{\sqrt{\frac{L^2}{4}+y^2}} - \frac{-1}{\sqrt{\frac{L^2}{4}+y^2}}\right)\hat{x} + \frac{kQ}{L}\left(\frac{L/2}{y\sqrt{\frac{L^2}{4}+y^2}} - \frac{-L/2}{y\sqrt{\frac{L^2}{4}+y^2}}\right)\hat{y} \\
&= \frac{kQ}{y\sqrt{\frac{L^2}{4}+y^2}}\hat{y}.
\end{aligned}$$

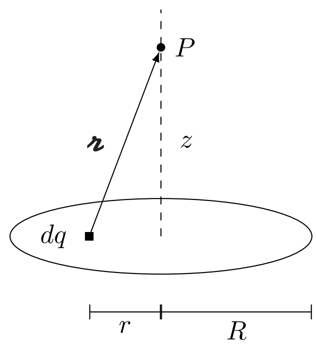
Note that the $\hat{x}$ term cancelled out. We could have seen that this would happen way back at the beginning because of symmetry, made a note of that in step 3, and only done one of the integrals to speed things up.

Solution 10.3: We use the same four steps as the problem above.

(a) Choose a coordinate system with the origin at the center of the ring. We will work in cylindrical coordinates with the z-axis perpendicular to the disk.

(b) Consider a chuck of the disk dq. Then $dq = \sigma dA = \sigma r dr d\theta$. (One can see that the area of a small section of a disk between r and $r + dr$ and between θ and $\theta + d\theta$ is change in radius $\times$ arclength $=$ $(r - r + dr) \times r d\theta = r dr d\theta$, so $dA = r dr d\theta$ in cylindrical coordinates.)

(c) We draw the vector $\boldsymbol{\mathscr{r}}$ from dq to the point we can about. See the picture.



In cylindrical coordinates,

$$\boldsymbol{\mathscr{r}}(r, \theta) = -r\hat{r} + z\hat{z}.$$

Therefore

$$d\boldsymbol{E}(r, \theta) = \frac{kdq}{\mathscr{r}^3}\boldsymbol{\mathscr{r}} = \frac{k\sigma r dr d\theta}{(r^2 + z^2)^{3/2}}\left[-r\hat{r} + z\hat{z}\right].$$

Before we integrate this, it pays to think for a second about symmetry. Because this is rotationally symmetric, the electric field will also be rotationall symmetric; the $\hat{r}$-component will cancel out when we integrate around the entire circle. Therefore we only have to worry about the z-component:

$$dE_z(r, \theta) = \frac{kq}{\mathscr{r}^3}\boldsymbol{\mathscr{r}} = \frac{kq}{(r^2 + z^2)^{3/2}} z.$$

(d) Integrating over θ and r,

$$\begin{aligned}
\boldsymbol{E} &= \int dE_z(r, \theta)\hat{z} \\
&= \int_0^{2\pi}\int_0^R \frac{k\sigma r dr d\theta}{(r^2 + z^2)^{3/2}} z\hat{z} \\
&= k\sigma z \left(\int_0^{2\pi} d\theta\right)\int_0^R \frac{r dr}{(r^2 + z^2)^{3/2}}\hat{z} \\
&= \frac{1}{4\pi\varepsilon_0}\sigma z(2\pi) \left.\frac{-1}{(r^2 + z^2)^{1/2}}\right|_0^R \hat{z} \\
&= \frac{\sigma}{2\varepsilon_0} z\left[\frac{-1}{\sqrt{R^2 + z^2}} - \frac{-1}{\sqrt{z^2}}\right]\hat{z} \\
&= \frac{\sigma}{2\varepsilon_0}\left[1 - \frac{z}{\sqrt{R^2 + z^2}}\right]\hat{z}.
\end{aligned}$$

We can easily take the limit $R \to \infty$, in which case

$$\lim_{R\to\infty} \frac{z}{\sqrt{R^2 + z^2}} = 0,$$

so $\boldsymbol{E} = \frac{\sigma}{2\varepsilon_0}\hat{z}$. Physically, this is the electric field due to an infinite plane of uniform charge. It makes sense that the z-distance has disappeared since, if you're looking at an infinite plane and you zoom in or out, everything looks exactly the same. This result will be very useful when we look at capacitors next week.

Solution 10.4: We assume the plates are modelled by infinite plates with uniform charge density σ. Then the electric field in the middle is

$$\boldsymbol{E} = 2\frac{\sigma}{2\varepsilon_0}\hat{x} = \frac{\sigma}{\varepsilon_0}\hat{x} = \frac{Q}{A\varepsilon_0}\hat{x}.$$

The force of the particles is then

$$\boldsymbol{F}_1 = -q\boldsymbol{E} = m\boldsymbol{a} \text{ and } \boldsymbol{F}_2 = q\boldsymbol{E} = M\boldsymbol{a_2}.$$

So one particle will accelerate right and the other will accelerate left giving equation of motion

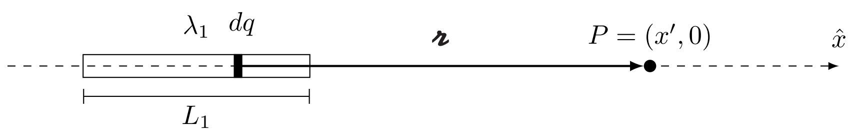
$$x_1(t) = D - \frac{1}{2}\frac{qE}{m}t^2 \text{ and } x_2(t) = \frac{1}{2}\frac{qE}{M}t^2$$

where $E = |E|$. One can solve $x_1(t^*) = x_2(t^*)$ to find t^* and from there, $x_{\text{meet}} = Dm/(m+M)$. Notice that the electric field is the same for both particles and therefore cancels out in the solution.

Solution 10.5:

(a) The first step is to compute the electric field due to one of the rods. We can then find the force on each part of the second rod due to the electric field of the first rod. We make the assumption that the rods do not feel their own electric fields. We also assume the rods are thin enough to be treated as lines.

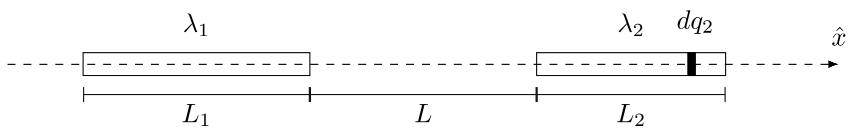
Think about just one rod at a time. Let's find the electric field as point P.



We choose coordinates with $x = 0$ at the left of the left-hand rod. Then we will integrate from $x = 0$ to $x = L_1$. The differential charge is $dq = \lambda_1 dx$ and $\boldsymbol{\mathscr{r}} = (x' - x)\hat{x}$. Then

$$\boldsymbol{E}(x') = \int d\boldsymbol{E} = \int_0^{L_1} \frac{kdq}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}} = \int_0^{L_1} \frac{k\lambda_1 dx}{(x'-x)^2}\hat{x} = k\lambda_1 \left.\frac{1}{x'-x}\right|_0^{L_1}\hat{x} = k\lambda_1\left(\frac{1}{x'-L_1} - \frac{1}{x'}\right)\hat{x}.$$

To find the force on rod 2, consider an infinitesimal chunk dq_2 of rod 2.



Then since $\boldsymbol{F} = q\boldsymbol{E}$ at each point on the second rod,

$$\begin{aligned}
\boldsymbol{F}_{12} &= \int \boldsymbol{E}(x')dq_2(x') \\
&= \int_{L_1+L_2+0}^{L_1+L+L_2} k\lambda_1\left(\frac{1}{x'-L_1} - \frac{1}{x'}\right)\hat{x}\lambda_2 dx' \\
&= k\lambda_1\lambda_2\Big(-\log x' + \log(x'-L_1)\Big)\Big|_{L_1+L_2+0}^{L_1+L+L_2}\hat{x}
\end{aligned}$$

using properties of logs gives

$$= k\lambda_1\lambda_2 \ln\left[\frac{(L_2+L)(L_1+L)}{L(L+L_1+L_2)}\right]\hat{x}.$$

(b) This is good practice at Taylor series and approximation. First we do some rearranging:

$$\ln\frac{(L_2+L)(L_1+L)}{L(L+L_1+L_2)} = \ln\frac{L\left(\frac{L_2}{L}+1\right)L\left(\frac{L_1}{L}+1\right)}{L^2\left(1+\frac{L_1}{L}+\frac{L_2}{L}\right)} = \ln\frac{1+\frac{L_1}{L}+\frac{L_2}{L}+\frac{L_1L_2}{L^2}}{1+\frac{L_1}{L}+\frac{L_2}{L}} = \ln\left[1+\frac{L_1L_2}{L^2\left(1+\frac{L_1}{L}+\frac{L_2}{L}\right)}\right]$$

Next we use the Taylor series for the natural log: $\ln(1+x) \approx x$ when $x \ll 1$. In this case this gives

$$\ln\left[1 + \frac{L_1 L_2}{L^2\left(1+\frac{L_1}{L}+\frac{L_2}{L}\right)}\right] \approx \frac{L_1 L_2}{L^2\left(1+\frac{L_1}{L}+\frac{L_2}{L}\right)}.$$

We can further approximate $1 + \frac{L_1}{L} + \frac{L_2}{L} \approx 1$, so

$$\frac{L_1 L_2}{L^2\left(1+\frac{L_1}{L}+\frac{L_2}{L}\right)} \approx \frac{L_1 L_2}{L^2}.$$

Therefore

$$\boldsymbol{F}_{12} = k\lambda_1\lambda_2 \ln\left[\frac{(L_2+L)(L_1+L)}{L(L+L_1+L_2)}\right]\hat{x} \approx k\lambda_1\lambda_2\frac{L_1L_2}{L^2}\hat{x} = k\frac{Q_1Q_2}{L^2}\hat{x}$$

where $Q_1 = \lambda_1 L_1$ and $Q_2 = \lambda_2 L_2$.

Solution 10.6:

(a) The net charge is $Q =$ arc length $\times$ linear charge density $= R\pi\lambda$.

(b) The force on the point charge is $\boldsymbol{F} = (-q)\boldsymbol{E}$. To find the electric field, we break the half-circle into chunks dq and integrate them. Taking a parameterization in terms of θ with $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, $dq = \lambda R d\theta$. Let the vector $\boldsymbol{\mathscr{r}}(\theta)$ go from dq to the center of the circle. In polar coordinates, $\boldsymbol{\mathscr{r}}(\theta) = R\hat{r}(\theta)$. Then

$$\boldsymbol{E} = \int_{-\pi/2}^{\pi/2} \frac{k dq}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}} = \int_{-\pi/2}^{\pi/2} \frac{k\lambda R\, d\theta}{R^2}\left[\cos\theta\hat{x} + \sin\theta\hat{y}\right].$$

(c) One can see from the symmetry of the situation that everything above and below will cancel out, so the force must point to the right.

(d) The integral is

$$\boldsymbol{E} = \frac{k\lambda}{R}\left(\int_{-\pi/2}^{\pi/2}\cos\theta\, d\theta\hat{x} + \int_{-\pi/2}^{\pi/2}\sin\theta\, d\theta\hat{y}\right) = \frac{2k\lambda}{R}\hat{x}.$$

Where in the last step one can either do the integrals or — better yet — use symmetry. Therefore the force is

$$\boldsymbol{F} = -\frac{2kq\lambda}{R}\hat{x}$$

Solution 10.7: This can either be done with a triple integral in cylindrical coordinates, or by integrating up the result of problem 1.3. Let's try the second method.

We can chop the cylinder up into disks of height dz running from $z = -h/2$ to $z = h/2$. The electric field at height $z_0 > h/2$ due to the disk at height z is

$$d\boldsymbol{E}(z_0; z) = \frac{\rho dz}{2\varepsilon_0}\left(1 - \frac{z_0 - z}{\sqrt{(z-z_0)^2 + r^2}}\right)\hat{z}$$

where the surface charge density is ρdz, which one can check has units of charge per unit area. Then the total electric field is

$$\boldsymbol{E}(z_0) = \int_{-h/2}^{h/2} d\boldsymbol{E}(z_0; z) = \int_{-h/2}^{h/2}\frac{\rho}{2\varepsilon_0}\left(1 - \frac{z}{\sqrt{(z-z_0)^2+r^2}}\right)dz\hat{z} = \frac{\rho}{2\varepsilon_0}\left(h - \sqrt{r^2 + (z_0 + \frac{h}{2}}^2 + \sqrt{r^2 + (z_0 - \frac{h}{2}}^2\right)\hat{z}.$$

This integral would be given in an exam.

Solution 10.8: There are two methods one could use here: take the solution for a ring of charge and integrate it vertically, or do it directly for the cylinder. I will opt to do the second.

Let's use cylindrical coordinates centered at the center of the cylinder. Consider a small chunk of charge of the cylinder at height ζ:

$$dq = \sigma\, dA = \sigma R d\theta dz.$$

Here $0 \le \theta \le 2\pi$, $-L/2 \le \zeta \le L/2$, and R is constant. The vector from dq to the point we care about, $(0, 0, z)$, is $\boldsymbol{\mathscr{r}} = -R\hat{r} + (z - \zeta)\hat{z}$. Therefore the electric field at $(0, 0, z)$ is

$$\begin{aligned}
\boldsymbol{E} &= \int_{-L/2}^{L/2}\int_0^{2\pi} \frac{kdq\boldsymbol{\mathscr{r}}}{\mathscr{r}^3} \\
&= \int_{-L/2}^{L/2}\int_0^{2\pi} \frac{k\sigma R\, d\theta\, dz}{(R^2 + (z-\zeta)^2)^{3/2}} \left[-R\hat{r} + (z-\zeta)\hat{z}\right]
\end{aligned}$$

By symmetry, the $\hat{r}$ component will cancel out as we integrate over θ, so we can drop it.

$$\begin{aligned}
&= \int_{-L/2}^{L/2}\int_0^{2\pi} \frac{k\sigma R\, d\theta\, dz}{(R^2 + (z-\zeta)^2)^{3/2}} \left[(z-\zeta)\hat{z}\right] \\
&= \frac{2\pi\sigma R}{4\pi\varepsilon_0} \int_{-L/2}^{L/2} \frac{(z-\zeta)\, dz}{(R^2 + (z-\zeta)^2)^{3/2}} \\
&= \frac{2\pi\sigma R}{4\pi\varepsilon_0} \left.\frac{-1}{\sqrt{R^2 + (z-\zeta)^2}}\right|_{-L/2}^{L/2} \\
&= \frac{\sigma R}{2\varepsilon_0}\left[\frac{1}{\sqrt{(z+L/2)^2 + R^2}} - \frac{1}{\sqrt{(z+L/2)^2 + R^2}}\right].
\end{aligned}$$

11 Gauss's Law

Gauss's law gives a shortcut for computing the electric field in situations with high symmetry. The statement of Gauss's law is, given a 3-dimensional surface S, then the flux of the electric field passing through the surface is equal to the change enclosed by the surface:

$$\int_S \boldsymbol{E} \cdot \hat{n} dA = \frac{Q_{\text{enc}}}{\varepsilon_0}. \tag{25}$$

The left-hand side of this equation is a surface integral; you should read up on surface integrals in a multi-variable calculus book at some point during this course. However, to do these problems the only fact you need to know is that the surface integral of unity is the surface area:

$$\int_S 1\, dA = \text{surface area of } S.$$

Gauss's Law is can be proved using Green's theorem and is true in all cases. *However*, Gauss's Law is only *useful to us* in certain situations. Usually we are given the location and strengths of charges and we want to find the electric field. Normally Gauss's law is no help in doing this, because all it gives is the integral of the electric field. But, in situations with high symmetry, then we can pull the electric field outside the integral:

$$\int_S \boldsymbol{E} \cdot \hat{n} dA = E \int_S dA = E \times \text{surface area of } S \quad \text{(only true for with lots of symmetry)}.$$

In this case, we can compute the charge enclosed and solve for E. There are three cases were we can do this:

1. Spherical symmetry. Then (in spherical coordinates) $\boldsymbol{E} = E(r)\hat{r}$.
2. Cylindrical symmetry. Then (in cylindrical coordinates)$\boldsymbol{E} = E(r)\hat{r}$.
3. Planar symmetry. Then (in Cartesian coordinates) $\boldsymbol{E} = E(z)\hat{z}$.

If we are in one of these special situations, we can use Gauss's law to determine $\boldsymbol{E}$ with the following procedure.

1. Identify the type of symmetry at play in this situation. Draw a "Gaussian surface", an invisible surface that respects the symmetry of the problem. For spherical symmetry, this is a sphere, etc.

2. Do the dot product inside the flux integral and show that, on each face of the surface, the dot product is either 0 or 1 and pull (a component of) the electric field outside the integral.

3. Compute the charge enclosed by the Gaussian surface.

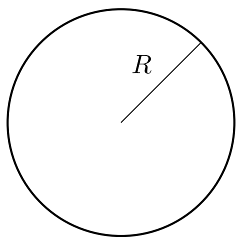
4. Solve for (a component of) the electric field.

These steps can actually be very quick; one you have practice at Gauss's law, you can solve problems with it in 4-5 lines.

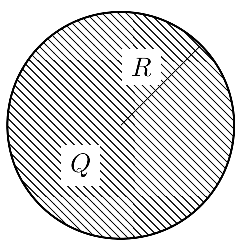
Let's dive into the problems now. As you practice these problems you'll get much quicker at them. To compensate for that, I've made each one uglier than the one before it.

11.1 Problems

Problem 11.1: Suppose there is a spherical shell (hollow ball) of radius R with charge Q spread uniformly on its surface. What is the surface charge density σ? What is the electric field $\boldsymbol{E}$ at all points in space? Keep in mind that $\boldsymbol{E}$ is a vector quantity.



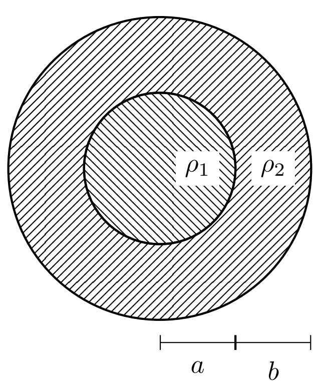
Problem 11.2: Suppose that there is a ball of radius R with charge Q uniformly distributed through the volume. What is the volume charge density ρ? What is the electric field $\boldsymbol{E}$ at all points in space?



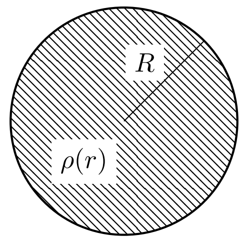
Problem 11.3: Suppose there is a ball of radius a with uniform volume charge density ρ_1. Suppose this is surrounded by a spherical shell of thickness b with uniform volume charge density ρ_2.

1. Write the volume charge density $\rho(r)$ as a piecewise function of r, the radial distance.

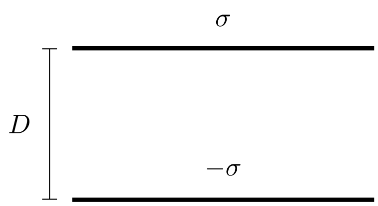
2. What is the electric field $\boldsymbol{E}$ at all points in space?



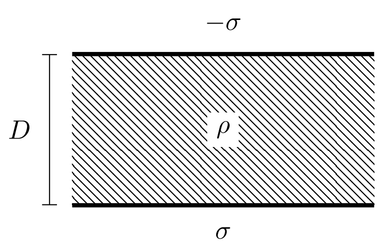
Problem 11.4: Suppose that there is a ball of radius R with non-uniform volume charge density $\rho(r)$ in spherical coordinates. What is the electric field at all points in space? Leave your answer in terms of $\rho(r)$.



Problem 11.5: Suppose there is an infinite plane of change of negligible thickness with a uniform surface change density σ and, a distance d away, there is another infinite plane with surface charge density $-\sigma$. Calculate $\boldsymbol{E}$ at all points in space.



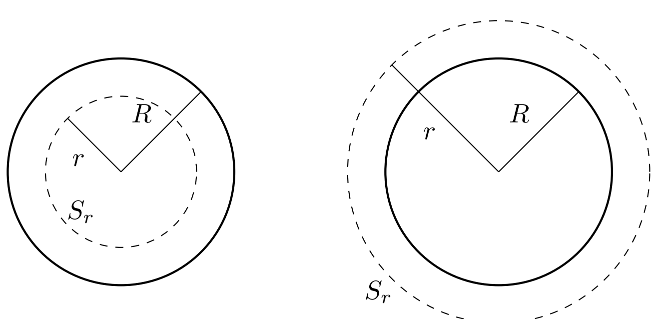
Problem 11.6: Suppose we have the same setup as the last problem, but add a slab of charge σ below the slab and a slab of charge $-\sigma$ above. Now what is $\boldsymbol{E}$ everywhere in space?



11.2 Solutions

Solution 11.1: Let's apply the four steps to find the electric field $\boldsymbol{E}$ at all points in space. There are two distinct cases: when $r < R$ and when $r > R$. We will therefore expect a piecewise function as answer.

1. This problem has spherical symmetry. Draw a Gaussian surface S_r, which is a sphere of radius r for some arbitrary r. There are two distinct cases: when $r < R$ and when $r > R$. We will therefore expect a piecewise function as answer.



$r < R$ $\qquad$ $r > R$

2. Due to the spherical symmetry, we know that the electric field has the form $\boldsymbol{E} = E(r)\hat{r}$. The outward unit normal vector to S_r is the radially outwards unit vector: $\hat{n} = \hat{r}$. Therefore

$$\int_{S_r} \boldsymbol{E}\cdot\boldsymbol{n}\, dA = \int_{S_r} E(r)\hat{r}\cdot\hat{r}\, dA = \int_{S_r} E(r) 1\, dA = E(r)\int_{S_r} 1\, dA = E(r) \times \text{surface area of } S_r = 4\pi r^2 E(r).$$

Here we were able to pull $E(r)$ outside the integral because the radius is a constant over S_r.

3. The charge enclosed by the Gaussian surface S_r is 0 when $r < R$ and Q when $r > R$. So

$$Q_{\text{enc}}(r) = \begin{cases} 0 & r < R \\ Q & r > R. \end{cases}$$

4. Gauss's law says

$$\int_{S_r} \boldsymbol{E} \cdot \hat{n} dA = \frac{Q_{\text{enc}}}{\varepsilon_0},$$

so using the result from parts 2 and 3 gives

$$4\pi r^2 E(r) = \frac{1}{\varepsilon_0} \begin{cases} 0 & r < R \\ Q & r > R. \end{cases}$$

so

$$E(r) = \frac{1}{4\pi\varepsilon_0} \begin{cases} 0 & r < R \\ Q & r > R. \end{cases}.$$

Remembering that $\boldsymbol{E} = E(r)\hat{r}$, the electric field at all points in space is

$$\boldsymbol{E} = \frac{Q}{4\pi\varepsilon_0} \begin{cases} 0 & r < R \\ 1 & r > R. \end{cases}$$

Physically, this means that a uniformly charged sphere looks like a point charge when you're outside it. When you're inside it, however, the electric fields cancel each other out and you see no field at all.

Problem 11.7: This exactly the same except the charge enclosed is now different. The volume charge density is

$$\rho = \frac{\text{charge}}{\text{volume}} = \frac{Q}{\frac{4}{3}\pi R^3}.$$

For $r > R$, the charge enclosed is Q. For $r < R$, things are more tricky. Since the charge is *uniformly* distributed, the charge inside S_r will be

$$Q_{\text{enc}} = \frac{\text{charge}}{\text{volume}} \times \text{volume} = \rho \frac{4}{3}\pi r^3 = \frac{Q}{\frac{4}{3}\pi R^3} \frac{4}{3}\pi r^3 = Q\frac{r^3}{R^3}.$$

Therefore

$$4\pi r^2 E(r) = \int_{S_r} \boldsymbol{E} \cdot \hat{n} dA = \frac{Q_{\text{enc}}}{\varepsilon_0} = \frac{1}{\varepsilon_0} \begin{cases} Q\frac{r^3}{R^3} & r < R \\ Q & r > R \end{cases}$$

so, solving for $E(r)$ and adding back in the unit vectors,

$$\boldsymbol{E} = E(r)\hat{r} = \frac{1}{4\pi\varepsilon_0} \begin{cases} \frac{1}{r^2} Q\frac{r^3}{R^3} & r < R \\ \frac{1}{r^2} Q & r > R \end{cases} \hat{r} = \frac{Q}{4\pi\varepsilon_0} \begin{cases} \frac{r}{R^3} & r < R \\ \frac{1}{r^2} & r > R. \end{cases} \hat{r}.$$

So the electric field increases linearly until it gets to the surface, and then falls off like $1/r^2$. From the outside, a spherical shell (the last problem) and a ball of charge (this problem) are indistinguishable.

Solution 11.2: The volume charge density is

$$\rho(r) = \begin{cases} \rho_1 & 0 < r < a \\ \rho_2 & a < r < b \\ 0 & r > b. \end{cases}$$

We can apply exactly the same argument as before to find the electric field; the only difference is the charge enclosed. For $0 < r < a$, this is the same as the last problem. For $a < r < b$, the charge enclosed is the charge inside radius a plus the charge between a and b:

$$Q_{\text{enc}} = \rho_1 \frac{4}{3}\pi a^3 + \rho_2 \left(\frac{4}{3}\pi r^3 - \frac{4}{3}\pi a^3\right) = (\rho_1 - \rho_2) \frac{4}{3}\pi a^3 + \rho_2 \frac{4}{3}\pi r^3.$$

When $r > b$, then the amount of charge is just the total charge:

$$Q_{\text{enc}} = (\rho_1 - \rho_2)\,\frac{4}{3}\pi a^3 + \rho_2 \frac{4}{3}\pi b^3$$

Therefore

$$\boldsymbol{E} = E(r)\hat{r} = \frac{1}{4\pi r^2}\frac{1}{\varepsilon_0}\begin{cases}\rho_1 \frac{4}{3}\pi r^3 & 0 < r < a\\ (\rho_1 - \rho_2)\,\frac{4}{3}\pi a^3 + \rho_2 \frac{4}{3}\pi r^3 & a < r < b\\ (\rho_1 - \rho_2)\,\frac{4}{3}\pi a^3 + \rho_2 \frac{4}{3}\pi b^3 & b < r\end{cases} = \frac{1}{3\varepsilon_0}\hat{r}\begin{cases}\rho_1 r & 0 < r < a\\ (\rho_1 - \rho_2)\frac{a^3}{r^2} + \rho_2 r & a < r < b\\ (\rho_1 - \rho_2)\frac{a^3}{r^2} + \rho_2 \frac{b^3}{r^2} & r > b.\end{cases}$$

Solution 11.3: This is really more of a calculus question than a physics question. What we're really asking for when we compute $Q_{\text{enc}}(r)$ is the volume integral over the ball of radius r, B_r:

$$Q_{\text{enc}} = \int_{B_r} \rho(r)\, dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{r} \rho(r) r^2 \sin\theta\, dr d\theta d\varphi = 4\pi \int_0^r \rho(r) r^2 dr.$$

Unfortunately, physicists and mathematicans often use different conventions for θ and φ in spherical coordinates. Physicists use θ as a polar angle running from 0 to π from the north pole down to the south pole and use φ as an azimuthal angle running from 0 to 2π around lines of latitude of the sphere. Mathematicians switch around θ and φ. Personally I think the math convention makes more sense because θ runs from 0 to 2π in polar coordinates and it should work the same way in spherical coordinates. However, all the physics books are written the other way, so there's not really any hope of changing it now.

Anyway, ranting aside, the electric field would be

$$\boldsymbol{E} = \frac{1}{4\pi\varepsilon_0 r^2} Q_{\text{enc}}\hat{r} = \frac{1}{4\pi\varepsilon_0 r^2} 4\pi \int_0^r \rho(r) r^2 dr \hat{r} = \frac{1}{\varepsilon_0 r^2}\int_0^r \rho(r') r'^2\, dr' \hat{r}.$$

I've been a bit sloppy through this question about integrating over r but also having r be the bound of integration. Really we should integrate over some dummy variable r' and then let 0 and r be the bounds of the integration. I've fixed this in the last step only.

Solution 11.4: There are several ways to do this problem

- Use the result from Exercise 2 of Section 1 above, take the limit $R \to \infty$ and then use superposition to find the electric field.
- Use Gauss's law to find the electric field for an infinite plane of uniform charge and then use superposition. We cannot use Gauss' law for both planes at once because, when you rotate everything by π, the situation is not the same because $\sigma \neq -\sigma$. Therefore we need to do the planes one at a time.

Let's do the first option because it's the easiest and you've probably seen the second one in the textbook or in class. From above we have that the electric field for an infinite plane of uniform charge density σ is

$$\boldsymbol{E} = \frac{\sigma}{2\varepsilon_0}\,\text{sgn}(z)\hat{z}.$$

If you haven't encountered it before, the "sign function" tells you what the sign of a quantity is:

$$\text{sgn}(x) = \begin{cases} +1 & x > 0\\ 0 & x = 0\\ -1 & x < 0.\end{cases}$$

In this case, this tells us the electric field points up on top of the plane and points down beneath it. This is in fact a general fact whenever there is planar symmetry: the electric field has the form $\boldsymbol{E} = E(z)\hat{z}$ where $E(z) = -E(-z)$. $E(z)$ is an odd function because, if you rotate the entire situation by π, then it must be the same.

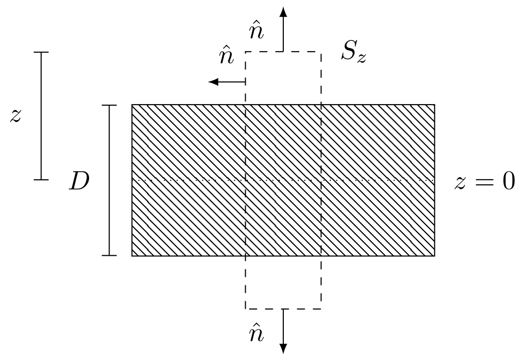
For both planes at once, we use superposition to find

$$\boldsymbol{E} = \frac{\sigma}{2\varepsilon_0}\,\text{sgn}\left(z + \frac{D}{2}\right)\hat{z} + \frac{-\sigma}{2\varepsilon_0}\,\text{sgn}\left(z - \frac{D}{2}\right)\hat{z} = \frac{\sigma}{\varepsilon_0}\hat{z}\begin{cases}1 & |z| < D/2\\ 0 & |z| > D/2.\end{cases}$$

So the electric field is constant and vertical inside and zero outside. We will use this as a model for a capacitor in the near future.

Solution 11.5: Again, we cannot directly apply Gauss's law because the sitaution is not symmetric under a flip $z \mapsto -z$. However, we can find the electric field for just the slab using Gauss's law and then add in the plane with superposition.

Image the Gaussian surface S_z, a cylinder with height $2z$ and area A on the top and bottom. We don't need a cylinder in particular; any surface with a straight edges whose top and bottom are flat and the same shape would work just as well. There are two cases: $z > D/2$ (shown in picture) and $z < D/2$.



First let's find the flux. The outward pointing unit normal to S_z points in different directions on the various faces of S_z. On the top it point in the $\hat{z}$ direction, on bottom it points in the $-\hat{z}$ direction and on the sides it points in the $\hat{r}$ direction, which is perpendicualr to $\hat{z}$. Therefore

$$\begin{aligned}
\int_{S_z} \boldsymbol{E}\cdot\hat{n}\,dA &= \int_{\text{top}} \boldsymbol{E}\cdot\hat{n}\,dA + \int_{\text{bottom}} \boldsymbol{E}\cdot\hat{n}\,dA \int_{\text{sides}} \boldsymbol{E}\cdot\hat{n}\,dA \\
&= \int_{\text{top}} E(z)\hat{z}\cdot\hat{z}\,dA + \int_{\text{bottom}} E(-z)\hat{z}\cdot(-\hat{z})\,dA \int_{\text{sides}} E(z)\hat{z}\cdot\hat{r}\,dA \\
&= E(z)\int_{\text{top}} dA - E(-z)\int_{\text{bottom}} +0
\end{aligned}$$

but $E(z) = -E(-z)$ and the surface area of the top and bottom A, so

$$= 2E(z)A.$$

Meanwhile the charge enclosed by S_z is $\rho(2z)A$ when $z < D/2$ and ρDA for $z > D/2$. Therefore

$$2E(z)A = \int_{S_z} \boldsymbol{E}\cdot\hat{n}\,dA = \frac{1}{\varepsilon_0}\begin{cases} \rho DA & 0 \le z > D/2 \\ \rho 2zA & z < D/2 \end{cases}$$

so the electric field for the slab is

$$\boldsymbol{E} = E(z)\hat{z} = \frac{1}{2A\varepsilon_0}\operatorname{sgn}(z)\hat{z}\begin{cases} \rho 2zA & |z| < D/2 \\ \rho DA & 0 \le |z| > D/2 \end{cases} = \frac{\rho}{\varepsilon_0}\operatorname{sgn}(z)\hat{z}\begin{cases} 2z & |z| < D/2 \\ D & |z| < D/2. \end{cases} \tag{26}$$

The electric field for the whole situation is therefore

$$\boldsymbol{E} = \boldsymbol{E}_{\text{slab}} + \boldsymbol{E}_{\text{plane}} = \frac{\rho}{\varepsilon_0}\operatorname{sgn}(z)\hat{z}\begin{cases} 2z & |z| < D/2 \\ D & |z| < D/2. \end{cases} + \frac{-\sigma}{2\varepsilon_0}\operatorname{sgn}\left(z + \frac{D}{2}\right)\hat{z}.$$

We can combine this into a piecewise function, but it's not particularly nice, so I haven't bothered.

Solution 11.6: We can get this just using superposition of the previous results:

$$\begin{aligned}
\boldsymbol{E} &= \boldsymbol{E}_{\text{slab}} + \boldsymbol{E}_{\text{top plane}} + \boldsymbol{E}_{\text{bottom plane}} \\
&= \frac{\rho}{\varepsilon_0}\operatorname{sgn}(z)\hat{z}\begin{cases} 2z & |z| < D/2 \\ D & |z| < D/2. \end{cases} + \frac{-\sigma}{2\varepsilon_0}\operatorname{sgn}\left(z + \frac{D}{2}\right)\hat{z} + \frac{\sigma}{2\varepsilon_0}\operatorname{sgn}\left(z - \frac{D}{2}\right)\hat{z}.
\end{aligned}$$

12 Gauss's Law With Conductors

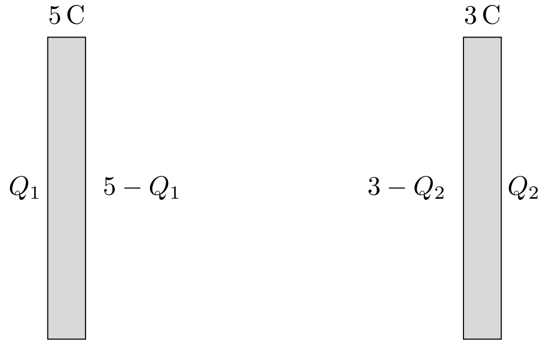
These problems are slightly different from Gauss's law problems you have seen before in that they involve conductors. When conductors are involved, one can use physical principles to determine the charge distributions, so they will not be specified in the problems. The following principles are useful:

1. The electric field inside a conductor is zero (in the static case).
2. Any net charge on a conductor is found on the surface (note that this can include both inner and outer surfaces if there are holes in the middle of the conductor!).
3. The electric field is always perpendicular to the surface of a conductor.

12.1 Problems

Problem 12.1: A pair of thick conducting slabs are fixed in plae near on another as shown, with their faces parallel. The faces have area A, which we will take to be very large compared to the slab's separation.

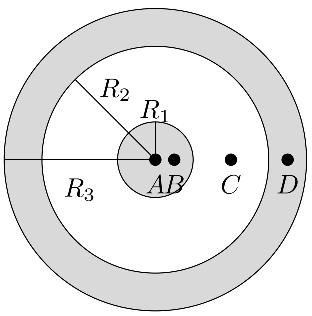
Initially the slabs are neutral, but then a net charge of $5\,\mathrm{C}$ is placed on the left slab, and a net charge of $3\,\mathrm{C}$ is placed on the right slab. When things have settled down, some amount of charge has migrated to the outer faces of the slab and some amount of charge has migrated to the inner faces. Of course, the charge on the left and right slabs must add up to $5\,\mathrm{C}$ and $3\,\mathrm{C}$ respectively.



1. Treating Q_1 and Q_2 as known quantities, use superposition to find an expression for the electric field within the left slab.
2. Taking into account the fact that the slab is a conductor, use this expression to find Q_1.
3. Perform a similar analysis to find Q_2.
4. Show that the charges on the inner surfaces of the slabs are equal and opposite.
5. Show that the charges will always be equal and opposite, regardless of how much charge is on either slab by drawing an appropriate Gaussian surface.

Source: Workbook page 64, problem 1.

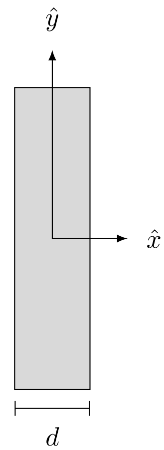
Problem 12.2: A long cylinder of radius R_1 is surrounded by a concentric cylindrical tube with inner radius R_2 and outer radius R_3. The inner cylinder is an insulator, and is uniformly charged with a charge density ρ. The outer cylindrical tube is a conductor with no *net* charge. (Cross-section shown.)



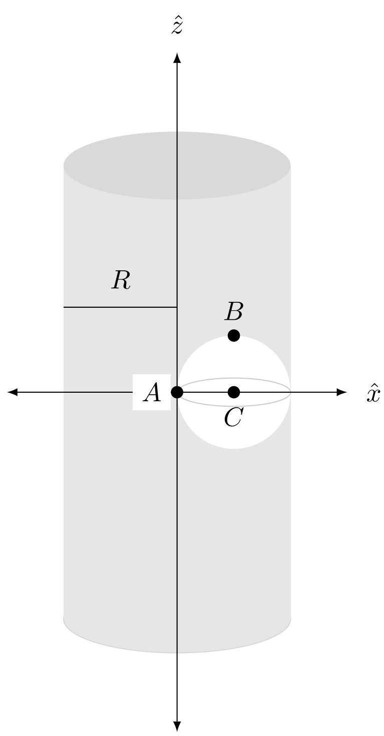
What is the magnitude of the electric field at points A, B, C, D, and E (as a function of r, the distance to the cylinder's central axis)? What is the surface charge σ_i on the inner surface of the conductor? What is the surface charge σ_o on the outer surface of the conductor?
(Modified from Giancoli 22.38.)

Problem 12.3: A flat slab of nonconducting material carries a uniform charge per unit volume ρ_E. The slab has thickness d which is small compared to the height and breadth of the slab. Determine the electric field as a function of x (a) inside the slab and (b) outside the slab (at distances much less than the slab's height or breadth). Take the origin at the center of the slab.

(*Giancoli 22.46*)



Problem 12.4: The figure below shows an infinitely long insulating cylinder with uniform charge density ρ and a radius R. Inside the cylinder is an empty spherical cavity with radius $R/2$. What is the electric field (as a vector!) at points A,B, and C?
(Speliotopoulous, Midterm 2, Fall 2012.)



12.2 Solutions

Solution 12.1:

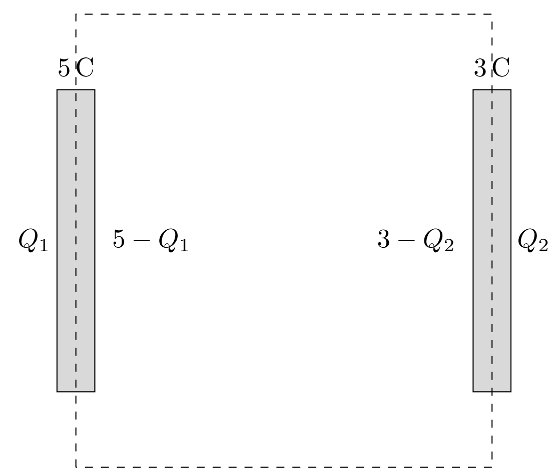
1. Recall that the electric field of a plate of charge is
$$E = \frac{\sigma}{2\varepsilon_0},$$
directed away from the plate. Therefore the field inside the left slab is
$$E = \frac{Q_1}{2\varepsilon_0 A} - \frac{5-Q_1}{2\varepsilon_0 A} - \frac{3-Q_2}{2\varepsilon_0 A} - \frac{Q_2}{2\varepsilon_0 A} = \frac{1}{2\varepsilon_0 A}\left[Q_1 - (5-Q_1) - (3-Q_2) - (Q_2)\right].$$

2. Since the slab is a conductor, the electric field must be zero inside it, so
$$0 = \frac{1}{2\varepsilon_0 A}\left[Q_1 - (5-Q_1) - (3-Q_2) - (Q_2)\right] = \frac{1}{2\varepsilon_0 A}\left[Q_1 - 5 + Q_1 - 3 + Q_2 - Q_2\right] = \frac{1}{2\varepsilon_0 A}\left[2Q_1 - 8\right] \implies Q_1 = 4.$$

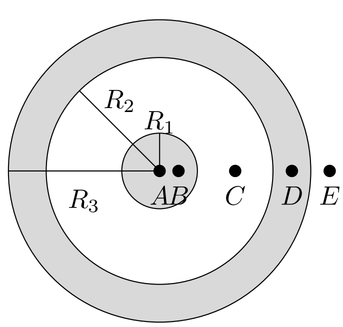
3. For the right-hand slab,
$$0 = E = \frac{1}{2\varepsilon_0 A}\left[Q_1 + (5-Q_1) + (3-Q_2) - Q_2\right] \implies 8 - 2Q_2 = 0 \implies Q_2 = 4.$$

4. The charge on the inner left hand slab is $5 - Q_1 = 1$ and the charge on the inner right hand slab is $3 - Q_2 = -1$, so the charges are indeed equal and opposite.

5. The following Gaussian surface encloses the charges on the inner surfaces of the slab, but its surface is everywhere perpendicular to the electric field. This means
$$0 = \int \boldsymbol{E}\cdot d\boldsymbol{A} = \frac{Q_{\text{enc}}}{\varepsilon_0} \implies Q_{\text{enc}} = Q_{\text{inner left}} + Q_{\text{inner right}} = 0.$$



Solution 12.2:



By symmetry, $\boldsymbol{E} = E(r)\hat{r}$. We can immediately say that $\boldsymbol{E}_A = 0$ because there it's the only symmetric possibility. Since D is inside a conductor, $\boldsymbol{E}_D = 0$ as well. At point B, we use Gauss's law for a (non-conducting) cylinder and find

$$\boldsymbol{E}_B = \frac{\rho r_B}{2\varepsilon_0}\hat{r}.$$

At point C, Gauss's law for a non-conducting cylinder (or, equivalently, a wire, since we're outside) gives

$$\boldsymbol{E}_C = \frac{\rho R_1^2}{2\varepsilon_0 r_C}\hat{r}.$$

The conductor has no electric field inside it, so it must have charge density

$$\sigma_I = -\frac{\rho R_1^2}{2R_2}$$

spread out over its inner surface to cancel out the $\boldsymbol{E}$-field. This means it must have an equal and opposite amount of charge on the outer surface, which gives a surface charge density of

$$\sigma_O = \frac{\rho R_1^2}{2R_3}.$$

Outside the conductor, at point E, we don't even see the conductor because the electric field resumes outside; the charge enclosed by a Gaussian surface of radius r_E is the same charge enclosed by a surface with radius r_C. Therefore

$$\boldsymbol{E}_E = \frac{\rho R_1^2}{2\varepsilon_0 r_E}\hat{r}.$$

Solution 12.3: See Midterm 2 Review, Gauss's Law exercise 6.

Solution 12.4: See `https://tbp.berkeley.edu/exams/4106/download/`, problem 4.

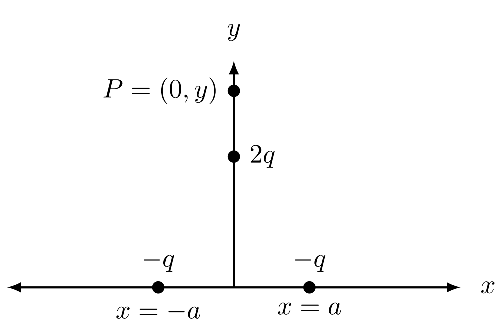
13 Electric Dipoles

13.1 Problems

Problem 13.1: Find an expression for the oscillation frequency of an electric dipole of moment $\boldsymbol{p}$ and moment of inertia I for small amplitudes of oscillaton about its equilibrium position in a uniform electric field $\boldsymbol{E} = E\hat{z}$. *Source: Halliday and Resnick 22.59*

Problem 13.2: Suppose that there was a dipole-like configuration where both charges were positive. Specifically, suppose charge Q was at the $x = -\ell/2$ and another charge Q was at $x = \ell/2$ on the x-axis. Show that the field given along an perpendicular bisector (the line $y = 0$) has magnitude $E = 2kQ/r^2$ far away from the charges. Why is $E \propto 1/r^2$ instead of $E \propto 1/r^3$ as it is for a normal dipole?
Source: Giancoli 21.64

Problem 13.3: Suppose there are three point charges arranged as follows.



The three charges form an equilateral triangle.

1. Find the position of the charge on the y-axis.

2. What is the electric field $\boldsymbol{E}$ at point $P = (0, y)$ on the y-axis due to the charge at $x = a$. Write your answer as a vector with both magnitude and direction in Cartesian coordintes (use $\hat{x}$ and $\hat{y}$).

3. What is the electric field at P from the charge at $x = -a$?

4. What is the electric field at P due to the charge on the y-axis?

5. What is the total electric field at P?

6. Suppose that P is very far from the origin. Then $\frac{a}{y} \ll 1$. Use a Taylor series to find the electric field to the first *non-zero* order in a/y.

7. What is the dipole moment of this configuration? Note that dipoles moments add as vectors.

Source: Lanzara, Midterm 2, 2014.

13.2 Solutions

Solution 13.1: An electric dipole in a uniform electric field feels a torque $\boldsymbol{\tau} = \boldsymbol{p} \times \boldsymbol{E}$. If the dipole makes an angle θ with the z axis, then it feels a torque $\tau = -pE\sin\theta$. From mechanics, we know that the analogue of Newton's second law for rotating objects is $\tau = I\frac{d^2\theta}{dt^2}$ where I is the moment of inertia. Then

$$I\frac{d^2\theta}{dt^2} = \tau = -pE\sin\theta \approx -pE\theta.$$

One can see this is directly analogous to

$$m\frac{d^2x}{dt^2} = F = -kx,$$

which gives rise to oscillations with frequency

$$\omega = \sqrt{\frac{k}{m}}.$$

In this case, a comparison of the equations show

$$\omega = \sqrt{\frac{pE}{I}}.$$

Solution 13.2: The electric field at a point $(0, y)$ is

$$\boldsymbol{E} = \frac{kQ}{\left(\frac{\ell^2}{4} + y^2\right)^{3/2}}\left[\frac{\ell}{2}\hat{x} + y\hat{y}\right] + \frac{kQ}{\left(\frac{\ell^2}{4} + y^2\right)^{3/2}}\left[-\frac{\ell}{2}\hat{x} + y\hat{y}\right] = \frac{2kQy\hat{y}}{\left(\frac{\ell^2}{4} + y^2\right)^{3/2}}.$$

In the limit where $y \gg \ell$,

$$\frac{1}{\left(\frac{\ell^2}{4} + y^2\right)^{3/2}} \approx \frac{1}{(y^2)^{3/2}} = \frac{1}{y^3},$$

so

$$\boldsymbol{E} \approx \frac{2kQ\hat{y}}{y^2}.$$

This decays like $1/r^2$ instead of $1/r^3$ because the charges add to first order (i.e. $1/r^2$) rather than cancelling.

Solution 13.3:

1. The triangle with vertices at $(a, 0), (0, 0)$ and the charge at $2q$ is a 30-60-90 triangle, so charge on the y-axis is at $y = \sqrt{3}a$.

2. We know the electric field is given by

$$\boldsymbol{E}_1(0, y) = \frac{k(-q)2q}{r^2}\hat{r}$$

where $\boldsymbol{r} = r\hat{r}$ is the vector from $(a, 0)$ to $P = (0, y)$. In coordinates, $\boldsymbol{r} = -a\hat{x} + \sqrt{3}a\hat{y}$. Therefore $r = |\boldsymbol{r}| = \sqrt{a^2 + y^2}$. Using the trick $\frac{\hat{r}}{r^2} = \frac{\boldsymbol{r}}{r}\frac{1}{r^2}$, we can write the electric field at P as as

$$\boldsymbol{E}_1(0, y) = \frac{k(-q)2q}{r^2}\frac{\boldsymbol{r}}{r^3} = -2kq^2\frac{-a\hat{x} + \sqrt{3}a\hat{y}}{(a^2 + y^2)^{3/2}} = 2akq^2\frac{\hat{x} - \sqrt{3}\hat{y}}{(a^2 + y^2)^{3/2}}.$$

3. This is the same, except we have $x = a \to x = -a$. In other words, we reflect our previous answer over the y-axis:

$$\boldsymbol{E}_2(0, y) = 2akq^2\frac{-\hat{x} - \sqrt{3}\hat{y}}{(a^2 + y^2)^{3/2}}.$$

4. This is a bit easier. Here $\boldsymbol{r} = \left|y - \sqrt{3}a\right|\hat{y}$, so

$$\boldsymbol{E}_3(0, y) = \frac{2kq^2}{(y - \sqrt{3}a)^2}\hat{y}.$$

5. The total electric field is then

$$\boldsymbol{E} = \boldsymbol{E}_1 + \boldsymbol{E}_2 + \boldsymbol{E}_3 = 2akq^2\frac{2\sqrt{3}a}{(a^2 + y^2)^{3/2}}\hat{y} + \frac{2kq^2}{(y - \sqrt{3}a)^2}\hat{y} = kq^2\left(\frac{4\sqrt{3}a}{(a^2 + y^2)^{3/2}} + \frac{2}{(y - \sqrt{3}a)^2}\right)\hat{y}.$$

Notice that the x-components of $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$ exactly cancelled because they are mirror images of one another.

6. This is really a question about dipoles in disguise and can be done in a single line in a week or two.

14 Electric Potential

The electric potential often provides a much easier way to compute quantities in electrostatics. Since the electric potential is a function on space — a scalar quantity — it is often quicker and easier to use than the electric field. Superposition also works for the electric potential.

14.1 Helpful Equations

$$\begin{aligned}
V(\boldsymbol{x}) &= \int_{\text{zero point}}^{\boldsymbol{x}} \boldsymbol{E} \cdot d\boldsymbol{\ell} && \text{(Potential from Electric Field)}\\
V(r) &= \frac{kq}{r} && \text{(Potential from a Point Charge)}\\
U &= qV && \text{(Potential Energy of a Point Charge)}\\
U &= \frac{\varepsilon_0}{2}\int_{\text{vol}} |\boldsymbol{E}|^2\, dV = \frac{\varepsilon_0}{2}\int_{\text{Vol}} \rho V\, dV = \frac{1}{2}\sum_{i \neq j}\frac{kq_iq_j}{r_{ij}} && \text{(Energy of a Charge Distribution)}\\
C &= \frac{Q}{V} && \text{(Defintion of Capacitance)}
\end{aligned} \tag{27}$$

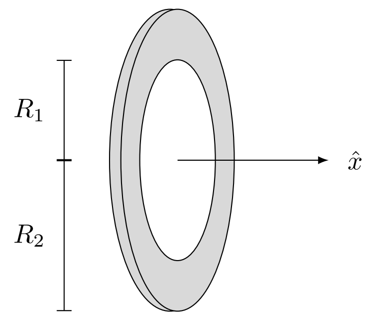
14.2 Problems

Problem 14.1: Three charges, $+5Q$,$-5Q$, and $+3Q$ are located on the y-axis at $y = +4a$, $y = 0$, and $y = -4a$, respectively. The point P is on the x-axis at $x = 3a$.

1. Draw a picture of the situation.
2. How much energy did it take to assemble these charges?
3. What are the x, y, and z components of the electric field $\mathbf{E}$ at P?
4. What is the electric potential V at point P, taking $V = 0$ at infinity?
5. A fourth charge of $+Q$ is brought to P from infinity. What are the x, y, and z components of the force $\mathbf{F}$ that is exerted on it by the other three charges?
6. How much work was done (by the external agent) in moving the fourth charge $+Q$ from infinity to P? This can be done without integrating anything!

(*MIT 8.02 Course Notes 3.10.9*)

Problem 14.2: A flat ring of inner radius R_1 and outer radius R_2 carries a uniform surface charge density σ. Determine the electric potential at points along the axis (the x-axis). Write the potential as a function of x. [*Hint*: Use superposition!]



(*Giancoli 22.35*)

Problem 14.3: A nonconducting sphere of radius r_0 carries a total charge Q distributed uniformly throughout its volume. Determine the electric potential as a function of the distance r from the center of the sphere for

1. $r > r_0$
2. $r < r_0$
3. Take $V = 0$ at $r = \infty$. Plot V versus r and E versus r.

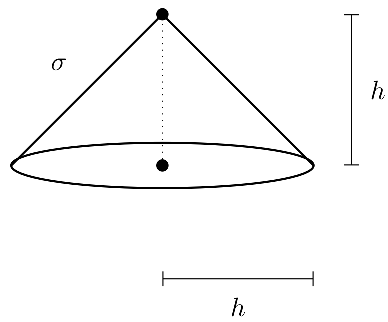
Source: Giancoli 23.19

Problem 14.4: A non-uniformly charges ring of radius R carries a linear charge density $\lambda(\theta) = \lambda_0 \cos^2 \theta$ where $\lambda_0 > 0$ and θ is the angle with the horizontal axis. The point P is distance z above the center of the ring.

1. Draw a picture of the situation and label it.
2. Calculate the electric potential at point P.

Souce: Modified from Bordel, Midterm 2, Spring 2013.

Problem 14.5: Suppose we have a charged surface that looks like an empty ice-cream cone (see picture). The height of the cone is h and the radius of the base is also h. The surface has a uniform surface charge density σ. Find the potential difference between the top of the cone and the middle of the base. Note that *only* the sloped surface of the cone is charged, *not* the base.



14.3 Solutions

Solution 14.1:

1. Sketch.
2. The total energy of a configuration of charges q_i where V_i is the potential from each charge is
$$U = \frac{1}{2}\sum_{i\neq j} q_i V_j.$$
3. Use $\boldsymbol{E} = \sum_i \boldsymbol{E}_i$.
4. Use $V(P) = \sum_i V_i(P)$.
5. This is just $U = QV(P)$.

Solution 14.2: Consider a chunk of charge dq at (r, θ). In polar coordinates, $dq = \sigma r dr d\theta$. This is a distance $\mathscr{r} = \sqrt{r^2 + x^2}$ away from the point we care about. Therefore

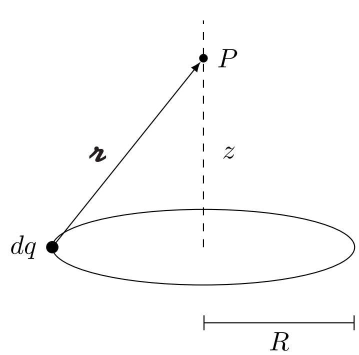
$$dV(r,\theta) = \frac{kdq}{r} = \frac{k\sigma r dr d\theta}{\mathscr{r}}.$$

So
$$V = \int_0^{2\pi}\int_{R_1}^{R_2} \frac{k\sigma r dr d\theta}{\mathscr{r}} = 2\pi k\sigma \int_{R_1}^{R_2} \frac{rdr}{\sqrt{r^2+x^2}} = 2\pi k\sigma\left[\sqrt{R_1^2+x^2} - \sqrt{R_2^2+x^2}\right].$$

Solution 14.3: Use Gauss's law and then $V(r) = -\int_\infty^r E(r')\,dr'$.

Solution 14.4:

1. Sketch.

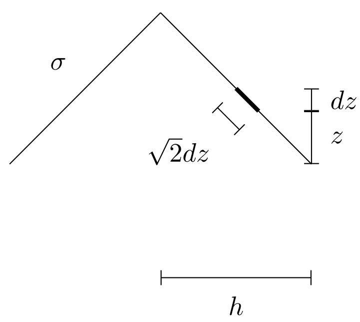


2. We will use superposition. Consider one small arc of the circle subtended by the angle $d\theta$. The infinitesimal charge on that arc is $dq(\theta) = \lambda(\theta)d\ell = \lambda_0 \cos^2\theta R d\theta$. Then

$$V(z) = \int_0^{2\pi} \frac{k\,dq}{\mathscr{r}} = \int_0^{2\pi} \frac{k\lambda_0 R\cos^2\theta d\theta}{\sqrt{R^2+z^2}} = \frac{k\lambda_0 R}{\sqrt{R^2+z^2}}\int_0^{2\pi}\cos^2\theta\, d\theta = \frac{k\lambda_0 R}{\sqrt{R^2+z^2}}\pi.$$

Solution 14.5: The tricky part here is thinking how to parametrize the sides of the cone. Because it's a 2D surface, we will need to do a double integral. Let use use two coordinates z and θ: z is the height from the base and θ is the polar angle around the cone.

Consider a segment of the cone from z to $z+dz$ and from θ to $\theta+d\theta$. From the side, this looks as follows.



Since the side of the cone makes an 45° angle with the base, the diagonal length along the patch dq must be $\sqrt{2}dz$. The other direction is $r(z)d\theta = (h-z)d\theta$ since the radius of the circle at height z is $h-z$. Overall, then,

$$dq = \sigma(h-z)\,d\theta\,\sqrt{2}\,dz.$$

The distance from the base to dq is $\sqrt{z^2+(h-z)^2}$ and the distance from dq to the top is $\sqrt{2}(h-z)$. Therefore the potential at the base is

$$V_{\text{base}} = \int_0^{2\pi}\int_0^h \frac{k\sqrt{2}\sigma(h-z)d\theta\,dz}{\sqrt{z^2+(h-z)^2}} = \frac{1}{2}hk\sigma\ln\frac{1+\sqrt{2}}{-1+\sqrt{2}}$$

The potential at the peak is similarly

$$V_{\text{peak}} = \int_0^{2\pi}\int_0^h \frac{k\sqrt{2}\sigma(h-z)d\theta\,dz}{\sqrt{2}(h-z)} = hk\sigma,$$

so the difference in potential is

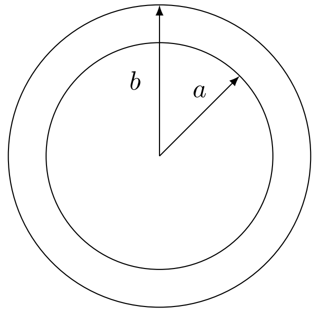
$$V_{\text{peak}} - V_{\text{base}} = hk\sigma - \frac{1}{2}hk\sigma\ln\frac{1+\sqrt{2}}{-1+\sqrt{2}} = \left(1-\frac{1}{2}\ln\frac{1+\sqrt{2}}{-1+\sqrt{2}}\right)hk\sigma \approx 0.118hk\sigma.$$

15 Capacitors

There are some very subtle types of problems involving removing dielectrics from capacitors. If the capacitor is hooked up to a battery which maintains a potential different V, then the situation grows quite complicated. The last few problems here address this situation and are quite challenging.

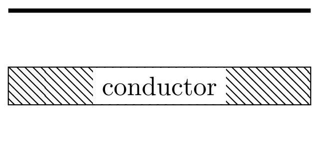
15.1 Problems

Problem 15.1: In this problem we will calculate the capacitance of a pair of cylinders of radii a and b and length ℓ. The picture shows a side-view.



1. Imagine placing charges $\pm Q$ on the cylinders and sketch the resulting electric field.
2. Calculate the strength of the electric field at points between the cylinders.
3. Find the difference in potential between the plates, by integrating the electric field $\boldsymbol{E}$.
4. What is the capacitance of the cylinders?
5. Calculate the energy stored in the capacitor using both $U = \frac{1}{2}CV^2$ and also by integrating the energy density $\mathcal{E} = \frac{\varepsilon_0}{2}|\boldsymbol{E}|^2$ over the volume between the places.

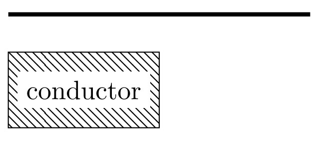
Problem 15.2: A parallel-plate capacitor is filled partially with a *conductor*.



What is the capacitance, assuming that the distance between the plates is d, the height of the conductor is x and the area of the plates is A?

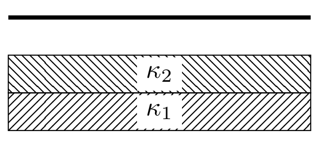
Calculate the energy stored in the capacitor using both $U = \frac{1}{2}CV^2$ and also by integrating the energy density $\mathcal{E} = \frac{\varepsilon_0}{2}|\boldsymbol{E}|^2$ over the volume between the places.

Problem 15.3: A parallel-plate capacitor is filled partially with a *conductor*.



Repeat the last question with the new shape of the conductor (area $A/2$, height x), making any approximations necessary.

Problem 15.4: A parallel-plate capacitor is filled with two different dielectrics as shown.



The distance between the plates is d, the area of each plate is A and the height of each dielectric is b.

1. Assume that the charge on the plates is $\pm Q$ and find $\boldsymbol{E}$ in each of the four regions.
2. Assuming $\kappa_1 > \kappa_2$, sketch the electric field in each region.
3. Find the potential difference V between the plates by integrating $\boldsymbol{E}$.
4. Compute the capacitance.
5. Find the capacitance by regarding this arrangement as four capacitors in series.

Problem 15.5: (Coaxial Capacitor) A capacitor consists of two coaxial cylinders of length L, with outer and inner radii a and b. Assume $L \gg a - b$, so that the end corrections may be neglected. Show that the capacitance is

$$C = \frac{2\pi\varepsilon_0 L}{\ln(a/b)}.$$

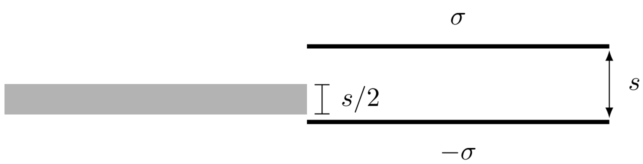
Verify that if the gap between the cylinders, $a - b$, is very small compared with the radius, this result reduces to the one that could have been obtained by using the formula for the parallel plate capacitor.

Source: Purcell 3.59.

Problem 15.6: The plaes of a parallel-plate capacitor have area A, separation x, and are connected to a battery with voltage V. While connected to the battery, the plates are pulled apart until they are separated by $2x$.

1. What are the initial and final energies stored in the capacitor?
2. How much work is required to pull the plates apart (assuming a constant speed)?
3. How much energy is exchanged with the battery?

Problem 15.7: (Sliding A Conductor Into A Capacitor)

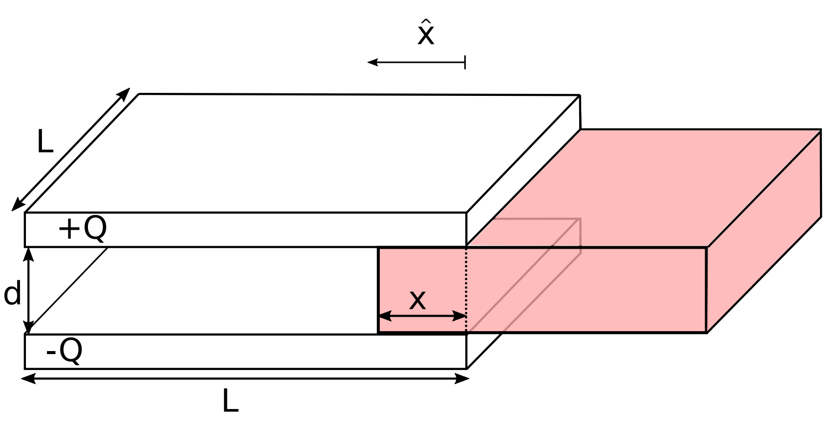


1. The plates of a capacitor have area A and separation s (assumed to be small). The plates are isolated, so the charges on them remain constant; the charge densities are $\pm\sigma$. A neutral con- ducting slab with the same area A but thickness $s/2$ is initially held outside the capacitor. The slab is released. What is its kinetic energy at the moment it is completely inside the capacitor? (The slab will indeed get drawn into the capacitor, as evidenced by the fact that the kinetic energy you calculate will be positive.)
2. Same question, but now let the plates be connected to a battery that maintains a constant potential difference. The charge densities are initially $\pm\sigma$. (Don't forget to include the work done by the battery, which you will find to be nonzero.)

Source: Purcell, 3.71.

Problem 15.8: A slab of width d and dielectric constant K is inserted a distance x into the space between two square parallel plates (with side length L) of a capactior as shown below. Determine the following, all as a function of x:

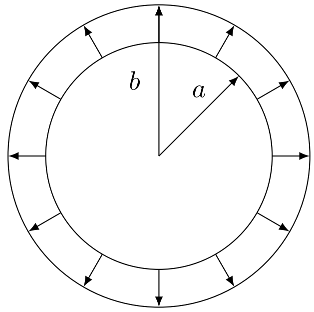
1. the capactance
2. the energy stored if the potential difference is maintained at V_0 by a battery
3. the magnitude and direction of the force exerted on the slab.



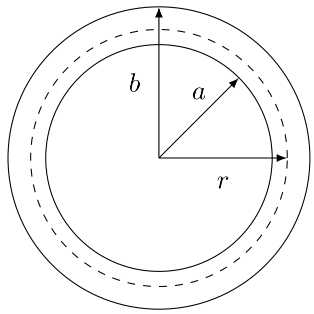
15.2 Solutions

Solution 15.1:

1. Sketch.



2. This is easily done by Gauss's law. We need only consider the inner cylinder. Imagine a cylindrical Gaussian surface with radius r such that $a < r < b$.



We make the approximation that the cylinder is so long that the ends can be neglected. Then the flux of the electric field will contribute only on the sides of the Gaussian cylinder, since we know the electric field is of the form $\boldsymbol{E} = E(r)\hat{r}$. Therefore

$$\int_{\mathrm{Cyl}_r} \boldsymbol{E} \cdot \hat{n} dA = E(r) \int_{\text{sides of the cylinder}} dA = E(r) 2\pi r \ell.$$

But Gauss's law says

$$\int \boldsymbol{E} \cdot \hat{n} dA = \frac{Q_{\mathrm{enc}}}{\varepsilon_0},$$

so

$$E(r) 2\pi r \ell = Q/\varepsilon_0 \implies \boldsymbol{E} = \frac{Q}{2\pi\varepsilon_0 r \ell}\hat{r}.$$

3. Integrating,

$$V = -\int_a^b \boldsymbol{E} \cdot \hat{r} dr = -\int_a^b \frac{Q}{2\pi\varepsilon_0 \ell} \frac{dr}{r} = -\frac{Q}{2\pi\varepsilon_0 \ell} \ln \frac{b}{a}.$$

However, we are interested in the magnitude of the potential different, so we discard the minus sign (this is what we always do for a capacitance problem).

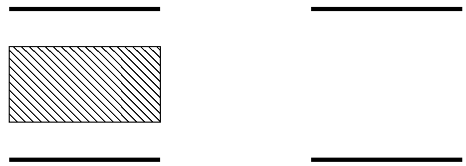
4. The capacitance is

$$C = \frac{Q}{V} = \frac{Q}{\frac{Q}{2\pi\varepsilon_0 \ell} \ln \frac{b}{a}} = \frac{2\pi\varepsilon_0 \ell}{\ln \frac{b}{a}}.$$

Solution 15.2: The conductor has zero electric field, so it is as if we had reduced the distance between the plates from d to $d - x$. Therefore the capacitance is

$$C = \frac{A}{d - x}\varepsilon_0.$$

Solution 15.3: One may approximate this as two capacitors in parallel, one with a conductor and the other without.



Applying the result of the previous problem, the capacitor on the left has capacitance

$$C_1 = \varepsilon_0 \frac{A/2}{d-x},$$

while the one on the right has

$$C_2 = \varepsilon_0 \frac{A/2}{d}.$$

Therefore, adding in parallel,

$$C = C_1 + C_2 = \varepsilon_0 \frac{A}{2}\left(\frac{1}{d-x} + \frac{1}{d}\right).$$

Solution 15.4:

1. For an ordinary parallel-plate capacitor, one applies Gauss's law to find that the electric field between the plates is

   $$\boldsymbol{E} = \frac{\sigma}{\varepsilon_0}\hat{z} = \frac{Q/A}{\varepsilon_0}\hat{z}$$

   where z points from the positive to the negative plate.

   Here, however, we have dielectrics. In a dielectric, the material polarizes in response to the electric field, creating a net dipole moment aligned with the electric field. The electric field due to all these dipoles points *against* the external field, reducing the electric field inside the metal. The dielectric constant κ says how much the field is reduced by: $\boldsymbol{E} \to \boldsymbol{E}/\kappa$ inside the material.

   The electric field in the four regions is then

   $$\boldsymbol{E} = \frac{Q/A}{\varepsilon_0}\hat{z}\begin{cases} 1 & \text{outside the dielectrics} \\ \frac{1}{\kappa_1} & \text{in the } \kappa_1 \text{ region} \\ \frac{1}{\kappa_2} & \text{in the } \kappa_2 \text{ region.} \end{cases}$$

2. Sketch.

3. The potential difference is

   $$V = \int_0^d \boldsymbol{E}(z)\cdot\hat{z}\,dz = \frac{Q/A}{\varepsilon_0}\left[\frac{1}{\kappa_1}\cdot b + \frac{1}{\kappa_2}\cdot b + (d-2b)\cdot 1\right].$$

4. The capacitance is therefore

   $$C = \frac{Q}{V} = \frac{A}{\frac{b}{\kappa_1} + \frac{b}{\kappa_2} + d - 2b}\varepsilon_0.$$

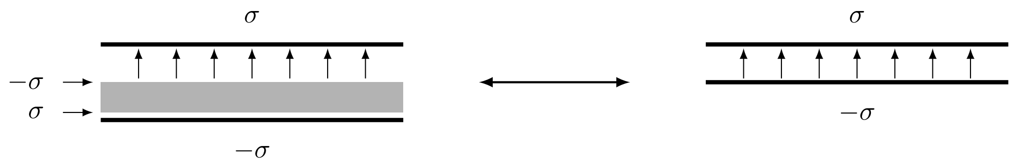
5. An alternative way to solve this is to think of as a bunch of capacitors in series. Then

   $$C = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \frac{1}{C_4}} = \frac{1}{\frac{1}{\frac{A}{d/2-b}\varepsilon_0} + \frac{1}{\kappa_1\frac{A}{b}\varepsilon_0} + \frac{1}{\kappa_2\frac{A}{b}\varepsilon_0} + \frac{1}{\frac{A}{d/2-b}\varepsilon_0}} = \frac{A}{\frac{b}{\kappa_1} + \frac{b}{\kappa_2} + d - 2b}\varepsilon_0.$$

Solution 15.5: See `https://tbp.berkeley.edu/exams/5037/download/`, Problem 4.

Solution 15.6: This is a remarkably subtle question that must be answered carefully. The reason *why* the conductor moves is because of the fringing fields of the capacitor attracting surface charges on the conductor. Since these are difficult to calculate, we will instead argue using energy methods.

1. Originally the capacitance of the system is $C = \varepsilon_0 \frac{A}{s}$. When the conductor is fully inserted, the effective distance between the plates is reduced by a factor of two. This is because the electric field inside a conductor must be zero, so there will be a surface charge σ on the bottom of the capacitor and $-\sigma$ on the top.



Therefore the electric field is only non-zero in the part of the capacitor *not* occupied by the conductor. The situation is therefore equivalent to a capacitor with half the distance between plates and no conductor. This would have a conductance of $C' = \varepsilon_0 \frac{A}{s/2} = 2C$. The plates are now effectively closer together so the capacitance is large.

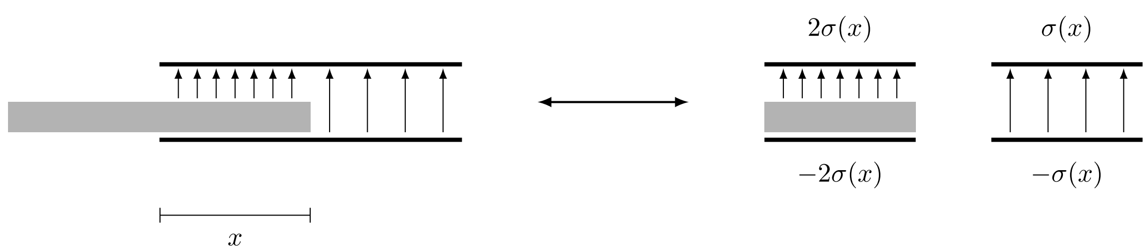
Since we know the conductor moves into the capacitor, we can use conservation of energy to determine the final kinetic energy of the conductor. Originally, all the energy is due to the capacitor, while at the end it is due to the capacitor and also the kinetic energy. Thus

$$\frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}\frac{Q^2}{C'} + K.$$

It is *very important* to use $\frac{1}{2}Q^2/C$ for the energy of the capacitor here, because the potential difference between the plates will *change* as the conductor is drawn inside. Therefore

$$K = \frac{1}{2}Q^2\left(\frac{1}{C} - \frac{1}{C'}\right) = \frac{Q^2}{2}\left(\frac{s}{\varepsilon_0 A} - \frac{1}{2}\frac{s}{\varepsilon_0 A}\right) = \frac{Q^2 s}{4\varepsilon_0 A} = \frac{1}{4}\frac{\sigma^2 A s}{\varepsilon_0}$$

2. First, let's try to understand why the conductor is pulled in here. When the capacitor is partially inserted, we can think of the situation as two capacitors in parallel:



To keep the potential difference constant, we will need to add more charge to the plates in the locations where we have the conductor. Since there is half as much distance, the field will need to be twice as strong, so for some of the area we will have charge density 2σ and the rest will have σ. The capacitor will still be pulled inside, because the battery does the work of maintaining the correct charges on the plates. As the conductor is moved completely inside, a total of σA is transferred from the bottom to the top plate by the battery.

Now let's calculate how much kinetic energy the conductor gains. To keep the signs straight here, it helps to take a thermodynamic approach. Suppose our system is the parallel plates and the charges on them but *not* the battery. Suppose the conductor is partially inserted a distance x.

Let us compare with the situation when it has slid in $x + dx$. The change in energy is

$$dE = đQ_{\text{heat}} + đW_{\text{on plates}} \implies dU_{\text{capacitor}} + dK = đW_{\text{by battery}}$$

(there is no heat in this situation). Integrating up,

$$U_{\text{cap,final}} - U_{\text{cap, initial}} + K = W_{\text{by battery}}$$

so

$$\frac{1}{2}C'V^2 - \frac{1}{2}CV^2 + K = QV.$$

Since capacitance depends only on geometry, C' is the same as the first part. Therefore

$$K = QV - \frac{1}{2}V^2\left(C - C'\right).$$

However, $V = \frac{Q}{C} = \frac{Qs}{\varepsilon_0 A} = \frac{\sigma s}{\varepsilon_0}$, hence

$$K = \sigma A\frac{\sigma s}{\varepsilon_0} - \frac{1}{2}\left(\frac{\sigma s}{\varepsilon_0}\right)^2\left(\frac{\varepsilon_0 A}{s} - 2\frac{\varepsilon_0 A}{s}\right) = \frac{\sigma^2 As}{\varepsilon_0} - \frac{1}{2}\frac{\sigma^2 As}{\varepsilon_0} = \frac{1}{2}\frac{\sigma^2 As}{\varepsilon_0}.$$

Solution 15.7: Thank you to Christian Schmidt for this extensive solution!

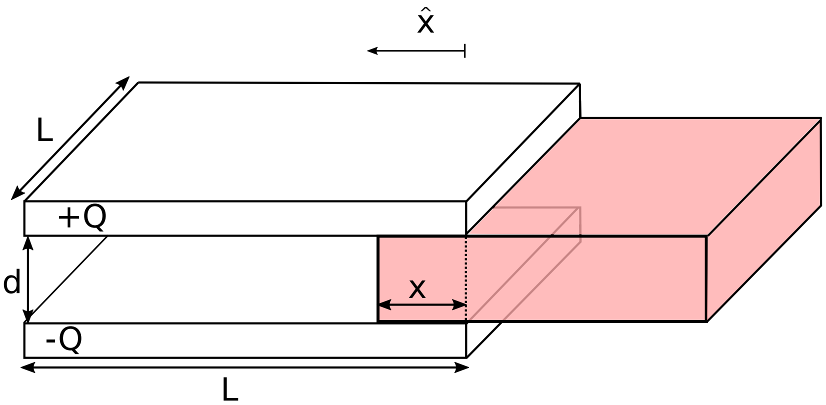


Figure 1: The physical setup of the problem

1. For the first part, we can assume that we know the formula for the capacity of a parallel plate capacitor with a vacuum between the plates, it's

$$C = \frac{\epsilon_0 A}{d}. \tag{28}$$

If we have a capacitor where the dielectric (with dielectric constant K) is fully inserted, the strength of the electric field between the plates is reduced, which increases the capacitance to

$$C = \frac{\epsilon_0 K A}{d}. \tag{29}$$

In this case, we have neither of these situations. But we can clearly split up the capacitor into two parts: The right part, where the dielectric is fully inserted, which has surface area $x \cdot L$, and the right part with no dielectric, which has surface area $(L - x) \cdot L$.

We should then notice that these two capacitors are connected to each other in parallel: The positively charged plate of the left capacitor is directly connected to the positively charged plate of the right capacitor, and the same thing holds for the negative plates. So we can add these capacitances:

$$C_{\text{total}} = C_{\text{left}} + C_{\text{right}}. \tag{30}$$

Stating (and explaining!) this gives you $\boxed{\text{2 points.}}$

Then writing down the individual capacities gives you $\boxed{\text{1 point each:}}$

$$C_{\text{left}} = \frac{\epsilon_0 L(L-x)}{d}, \quad C_{\text{right}} = \frac{\epsilon_0 K L x}{d}. \tag{31}$$

Then you get $\boxed{\text{1 point}}$ for writing down the sum

$$C(x) = \frac{\epsilon_0 L}{d}(L + (K-1)x). \tag{32}$$

2. Here we have to use the formula for the electrostatic energy stored in a capacitor

$$U = \frac{1}{2}C(x)V_0^2. \tag{33}$$

This is equivalent $U = \frac{Q^2}{2C}$, if we then plug in the right value for Q, $Q = CV_0$. This formula gives you $\boxed{\text{3 points}}$. We then have to plug in C to get

$$U = \frac{V_0^2 \epsilon_0 L}{2d}(L + (K-1)x), \tag{34}$$

which gives you $\boxed{\text{2 points}}$ if you wrote it down in the right variables.

3. This part is more involved than the others, and unfortunately, it's very easy to get the 'right' magnitude by completely wrong arguments. So if you didn't get points for this part, try to understand the actual solution. Let's first understand what's physically going on:

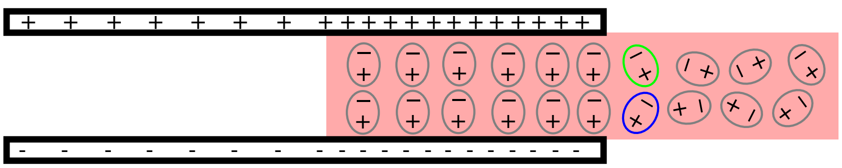


Figure 2: A 'microscopic' picture of the dielectric and the plates

There are two things we should observe in figure 3: The first one is that the dipoles which are just outside of the plates feel the fringe electric fields at the boundary. From the picture, it should be clear that the dipole I painted in green feels a force pulling it to the upper plate, and the dipole in blue feels a force to the lower plate. In total, the vertical components cancel out, and there is a net force which pulls the dielectric slab **to the left**. For finding this direction, either by using this argument or via the actual calculation we do below, you get $\boxed{\text{5 points}}$.

The second thing in the picture is that the charge density on the plates is not constant when we insert the dielectric. If we didn't keep the voltage difference between the plates constant, this would simply mean that some charges are pulled from the left parts of the plates to the right parts during the insertion, which reduces the potential difference. (Alternatively, note that $Q = CV$, if C increases and Q is constant, V has to decrease).

In this case, however, the voltage is constant, so there can be no charges leaving the left parts of the plates. So there have to be some charges added to the metal plates during the insertion. Alternatively, observe that if $Q = CV$ and C increases while V is constant, Q has to increase.

So in summary there has to be some voltage source connected to the plates that pushes an additional charge $\Delta Q(x) = (C(x) - C(0))V_0$ onto them – for example, a battery.

To find the magnitude of the force on the dielectric, we note that x describes the position of the slab in a coordinate system where the x-axis goes to the left (cf. figure 15.2). So we can find the force on the slab by the formula

$$\vec{F} = -\vec{\nabla}U_{\text{total}}, \tag{35}$$

where U_{total} is the total potential energy of the system. (Note that actually, x would have to be the center-of-mass coordinate of the slab, but because this is just a constant shift, this doesn't matter). So let's first find the total energy of the system (which consists of the battery and the capacitor):

$$U_{\text{battery}} = U_0 - \text{work} = U_0 - V_0\Delta Q(x) = U_0 - V_0^2(C(x) - C(0)), \tag{36}$$

where U_0 is the energy of the battery before the insertion. So

$$U_{\text{total}}(x) = \frac{1}{2}C(x)V_0^2 + \text{const.} - C(x)V_0^2 = \text{const.} - \frac{1}{2}C(x)V_0^2, \tag{37}$$

where the terms not depending on x are summed in the const. term. Plugging this into 35 gives

$$\vec{F} = \frac{V_0^2 \epsilon_0 L}{2d}(K-1)\hat{x}, \tag{38}$$

so since $K > 1$ we can also see here that the force goes to the left (note the direction of $\hat{x}$ in figure 15.2). Getting the right magnitude (in the right way, so by including the battery) gives you $\boxed{\text{5 points}}$. Note that if we had forgotten the battery, we would have, by sheer coincidence, gotten the same magnitude (and the wrong direction). So if you did the same derivation, but just forgot the battery and did everything else right, you got $\boxed{\text{2 points}}$.

As a last point, note that there's something obviously wrong about this answer: When x approaches L, so the slab is almost fully inserted, we would still have a non-zero force! In the physical picture, we can see that there are almost no dipoles left to be pulled in, so the force should go to zero. But when deriving the capacitance $C = \frac{\epsilon_0 A}{d}$, you assumed that the plates are so large that you can neglect any boundary effects, which you would have to take into account here. Fortunately, this only changes the result when the dielectric is not inserted at all or when it's pulled in almost all the way.

16 Magnetic Forces

We are now starting magnetism. Magnetism is always a bit more *odd* that electricity — there are a large number of effects that defy intuition and expectations. That, together with the large number of cross-products involved in magnetism formulas, make this part of the course somewhat more conceptually difficult than electricity. Most of the mathematical tools — integrating vectors or vector field over lines, surfaces, and volumes — are the same, but they're applied a little differently.

The goal this week will be to gain some intuition for how magnetic forces work and how to use superposition with magnetism.

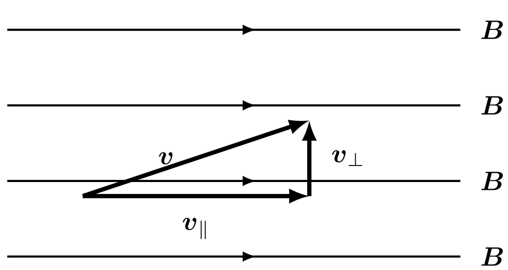
16.1 Helpful Equations

$$\boldsymbol{F} = q\boldsymbol{v} \times \boldsymbol{B}. \qquad \text{(Coulomb's Law)}$$

$$d\boldsymbol{F} = I d\hat{\ell} \times \boldsymbol{B} \qquad \text{(Infinitesimal Coulomb's Law for a Wire)}$$

16.2 Problems

Problem 16.1: (Charged particle in a uniform magnetic field) A charged particle with charge q and initial velocity $\boldsymbol{v}$ moves in a uniform magnetic field $\boldsymbol{B}$. If the angle between $\boldsymbol{v}$ and $\boldsymbol{B}$ is θ, describe the motion of the particle in the three cases specified in the figure below.



Problem 16.2: (Cyclotron Motion) Suppose there is a constant magnetic field $\boldsymbol{B} = -B_0\hat{z}$ everywhere in space. Suppose that a particle starts with velocity $\boldsymbol{v}_0 = v\hat{x}$ at time zero. This particle will undergo uniform circular motion.

1. Find the initial force on the particle. Which way does the particle bend?

2. Draw the particle's path and label the magnetic force along it periodically.

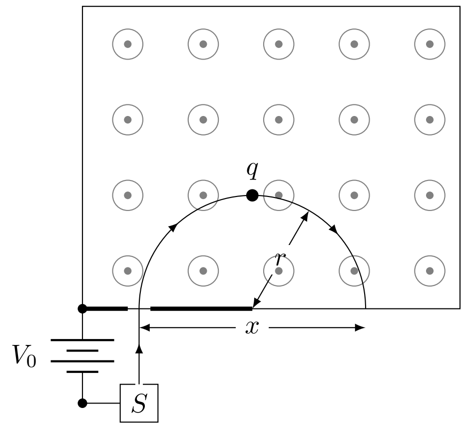
3. Since the particle undergoes uniform circular motion, it "orbits" around a certain point. How far away is the particle from this point? This distance is called the *cyclotron radius*. What is the frequency with which the particle goes around the circle? Remember than angular frequency is $\omega = v/r$.

Problem 16.3: (Mass Spectrometer) A Mass Spectrometer is a device that measures the mass of small charged particles. Often it is used to determine the composition of a mixture of unknown mixture of molecules. It works as follows: the molecules are ionized. A potential difference is then used to accerate the molecules and give them a velocity. The molecules then enter a chamber with a constant magnetic field, which causes them to curve. The radius of curvature depends on the mass of the particles; more massive particles will move further. By measuring where the particles strike the edge of the chamber, the mass of the particles can be determined. Let's calculate how this works.

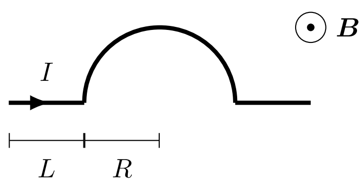
An ion of mass m and charge $+q$ is produced at rest in source S and accelerated across a potential difference V_0 and enters a magnetic field B as shown below. The ion moves in a semicircular path and strikes a plate at a distance x from the entrance slit. Show that the mass of the ion can be found by the following equation:

$$m = \frac{B^2 q}{8V_0} x^2.$$

Source: Modified from Halliday, Chapter 34-4, problem 22.

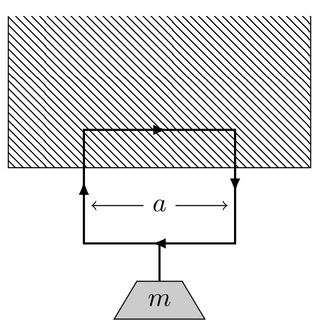


Problem 16.4: (Magnetic force on a wire) Find the net magnetic force on the whole wire of current I which consists of two straight segments of length L and a half circular segment of radius R as shown below. The magnetic field is constant and in the $\hat{z}$ direction.

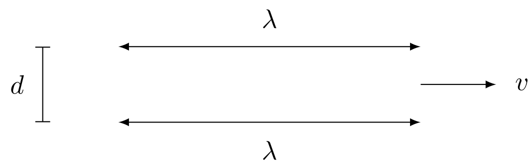


Problem 16.5: (Magnetic Suspension) A rectangular loop of wire of width a, supporting a mass m, is placed with one end in a uniform magnetic field $\boldsymbol{B}$, which points into the page in the shaded region of the figure.

1. For what current I in the loop would the magnetic force upwards exactly balance gravity downwards?
2. Now suppose we increase the current. Then the magnetic force up exceeds gravity and the weight is lifted. But *magnetic fields do no work*, so how does this happen? Clearly, *something* must be doing work on the mass — what is it?

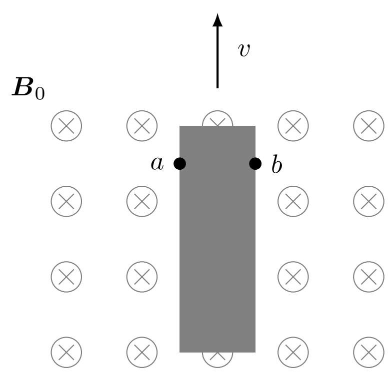


Problem 16.6: (Lorentz force) Two infinitely long wires, separated by a distance d are moving at velocity v toward the right. They both carry uniform linear charge density λ. (This problem has many parts, but each should be fairly short.)



1. Derive the electric field at the second wire produced by the first wire.
2. Find the electric force per unit length on the second wire.
3. Find the current produced by the first wire.
4. Derive the magnetic field at the second wire produced by the first wire.
5. Find the magnetic force per unit length on the second wire.
6. At what value of v will the electric and magnetic forces have the same magnitude? (Ignore any special relativistic effects.) Find the *numerical* value of this speed in meters per second to 3 decimal places. Where have you seen this value before?

Problem 16.7: (Hall effect) A metal strip of length L and width w and thickness t is moves with a velocity v through a magnetic field B_0 into the page as shown below. If a potential difference of V_0 is measured between points a and b across the strip, calculate the speed v of the strip.*(modified from Halliday, Chapter 34-4, problem 39)*



Problem 16.8: (Conducting Wire in nonuniform magnetic field) In a region of space, there is a magnetic feld $B = B_0(x\hat{x} + z\hat{z})$. A wire of length L carries a current I. One end of the wire is at the origin, and the wire makes an angle θ with the x axis, in the $x - y$ plane. What is the force on the wire if the current is directed away from the origin?

Problem 16.9: Suppose we have a rectangular block with length ℓ, width d, and heigh h in the x,y, and z directions respectively. A current $\boldsymbol{I} = I\hat{x}$ flows through it. There is an external magentic field $\boldsymbol{B} = B\hat{y}$.

1. Derive the Hall voltage V_H for this setup.
2. Find an expression for the Hall constant
$$K_H = \frac{V_H}{BI}$$
in terms of the charge carrier concentration n.
3. Using these results, how can you determine the sign and density of charge carriers in a material?

16.3 Solutions

Solution 16.1:

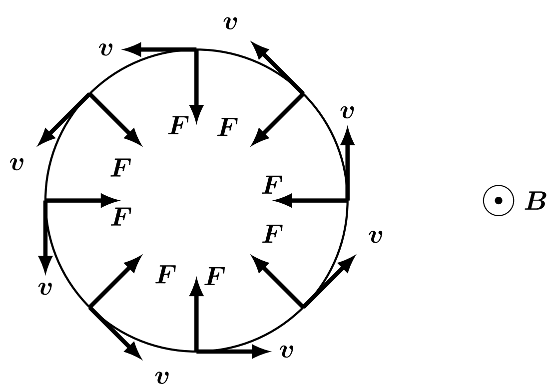
1. In case one, $\theta = 0$. Then $\boldsymbol{F} = q\boldsymbol{v} \times \boldsymbol{B} = 0$.
2. In case two, $\theta = \pi/2$. Then $\boldsymbol{F} = q\boldsymbol{v} \times \boldsymbol{B} = -qVB\hat{z}$, i.e. the force will be directed into the page.
3. In case two, $0 < \theta < \pi/2$. Then we can split up v as $\boldsymbol{v} = v\cos\theta\hat{x} + v\sin\theta\hat{y}$. We can then apply each of the two previous cases and find that $\boldsymbol{F} = q\boldsymbol{v} \times \boldsymbol{B} = -qvB\sin\theta\hat{z}$.

Solution 16.2:

1. The force on the particle initially is

$$\boldsymbol{F} = q\boldsymbol{v}_0 \times \boldsymbol{B} = qv\hat{x} \times (-B_0\hat{z}) = qvB_0\hat{y}.$$

2. See sketch. Let the z direction be towards us.



3. Since the particle undergoes uniform cicular motion, we know that the magnetic force must act as a centripedal force. Thus, choosing polar coordinates with the origin at the center of the circle,

$$-\frac{mv^2}{r}\hat{r} = \boldsymbol{F}_{\text{centripedal}} = \boldsymbol{F}_{\text{magnetic}} = -qvB\hat{r}.$$

Equating components gives $\frac{mv^2}{r} = qvB$. Solving for r gives us

$$r_{\text{cyclotron}} = \frac{mv}{qB}.$$

Therefore the angular frequency of the motion is

$$\omega = \frac{v}{r} = \frac{qB}{m}.$$

Solution 16.3: Once the ion enters the area of uniform magnetic field, it will undergo uniform cicular motion until it crashes into the wall. Using the formulas from the last problem, we can immediately see that

$$x = 2r_{\text{cyclotron}} = 2\frac{mv}{qB}.$$

We will find v using conversation of energy. As the ion travels across the potential difference V_0, conservation of energy implies

$$qV_0 = \frac{1}{2}mv^2 \implies v = \sqrt{\frac{2qV_0}{m}}.$$

Therefore

$$x = \frac{2m}{qB}\sqrt{\frac{2qV_0}{m}} \implies x^2 = \frac{4m^2}{qB}\frac{2qV_0}{m} \implies m = \frac{qB^2}{8V}x^2.$$

Mass spectrometers are used in chemistry to determine the components of chemical samples by weight. They are also used in nuclear physics to separate different isotopes of an element.

Solution 16.4: We will use our old friend superposition to solve this problem. Choose coordinates where the origin is at the center of the semicircle, and $\hat{x}$ is to the right.

On the two straight edges, we can immediately apply

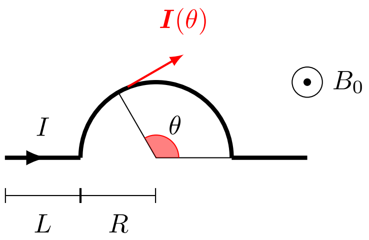
$$\boldsymbol{F} = \boldsymbol{I}L \times \boldsymbol{B} = -ILB\hat{y}.$$

Keep in mind there are two such contributions from the left and right parts of the wire.

For the loop of wire, we need to do something rather more involved. We will parameterize the circle by θ and integrate up the forces along it using

$$d\boldsymbol{F}(\theta) = \boldsymbol{I}(\theta) \times \boldsymbol{B} d\ell.$$

Consider an arbitrary point on the circle. See the picture.



We can see from the geometry that

$$d\boldsymbol{F}(\theta) = \boldsymbol{I}(\theta) \times \boldsymbol{B} d\ell = I\left(-\hat{\theta}\right) \times (-B\hat{z})\, d\ell = -IB\hat{r}d\ell = -IB\hat{r}\, Rd\theta.$$

Since the force points inwards all along the circle, we can see that the horizontal components will cancel out and the vertical components will add. Therefore the net force will be downwards, we need only consider

$$dF_y(\theta) = \hat{y} \cdot d\boldsymbol{F}(\theta) = -IB\hat{y} \cdot \hat{r}Rd\theta = -IB\sin\theta R\, d\theta.$$

So, noting that the current flows from π to 0,

$$\boldsymbol{F}_{\text{semi-circle}} = F_y\hat{y} = \int_\pi^0 -IB\sin\theta R\, d\theta\hat{y} = -IBR\cos\theta|_\pi^0\,\hat{y} = -IBR\left(\cos 0 - \cos\pi\right)\hat{y} = -2IBR\hat{y}.$$

Putting it all together,

$$\boldsymbol{F} = -ILB\hat{y} - 2IBR\hat{y} - ILB\hat{y} = -I\left(2L + 2R\right)B\hat{y}.$$

It is interesting to note that the semi-circle completely dropped out — it looks like it was just a straight wire of length $2L + 2R$. Why was this? Can you show that only the x-length matters for a wire with an arbitrary curve in the same constant magnetic field?

Solution 16.5:

1. There is a magnetic force on the portion of the wire inside region with magnetic field. The force on the left part of the wire is to the left; the force on the right part of the wire is to the right; the force on the top part of the wire is up. The left and right parts cancel each other out, so we only have to worry about the top. The force there is

$$\boldsymbol{F} = IaB\hat{z}.$$

   To exactly balance gravity, we need

$$IaB\hat{z} = mg \implies I = \frac{mg}{aB}.$$

2. Questions involve work done by magnetic fields are often surprisingly subtle. The fact that the magnetic field itself does no work is often misleading. Often times, the magnetic field will serve to *redirect* the motion of some other particles to do work. On might say that magnetic forces do work indirectly.

   In this case, when the loop starts to rise, the current will acquire a small y-component as the charges move upwards. The magnetic force will tilt to the left to accommodate this, ensuring that the magnetic force is perpendicular to the current. This means that the magnetic force will push *against* the charges in the wire, requiring more work *from the battery* (or whatever is driving the current) to overcome it. So the battery is what really does the work, and the magnetic field only redirects it.

   For a more in-depth discussion, see Griffith's *Introduction to Electrodynamics*, chapter 5.1.

Solution 16.6:

1. We can use Gauss's law with a Cylindrical Gaussian Surface of length L to find (with contributions only from the sides of the cylinder)

$$\int_S \boldsymbol{E} \cdot d\boldsymbol{A} = E(r) 2\pi r L = \frac{Q_{\text{enc}}}{\varepsilon_0} = \frac{\lambda L}{\varepsilon_0} \implies E(r) = \frac{1}{2\pi\varepsilon_0} \frac{\lambda}{r},$$

so the strength of the electric field at the second wire due to the first wire is $\frac{1}{2\pi\varepsilon_0}\frac{\lambda}{d}$ pointing away from the other wire.

2. The force per unit length is

$$\frac{F}{L} = \frac{\lambda^2}{2\pi\varepsilon_0 d},$$

directed away from the other wire.

3. Current is the amount of charge passing a given point per unit time:

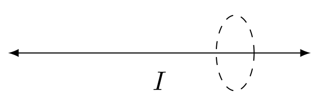
$$I = \frac{dQ}{dt} = \frac{dQ}{dx}\frac{dx}{dt} = \lambda v,$$

moving to the right.

4. For this part, one may just quote the result from the book, derive it from infinitesimal currents, or use Ampère's law. Let's use Ampère's law, which says the line integral of the magnetic field around a loop is equal to the current that flows through the loop:

$$\oint_C \boldsymbol{B} \cdot d\boldsymbol{\ell} = \mu_0 I_{\text{enc}}.$$

This is essentially the magnetic analogue of Gauss's law and, in situations with high symmetry, we can use it to *find* the magnetic field. Due to the cylindrical symmetry of this situation, we know $\boldsymbol{B} = B(r)\hat{\theta}$. Let us therefore take an Amperian loop which is a circle of radius r around the wire.



Then the current enclosed by the loop is just I, so

$$\oint_C \boldsymbol{B} \cdot d\boldsymbol{\ell} = \int B(r)\hat{\theta} \cdot \hat{\theta} d\theta = B(r) 2\pi r = \mu_0 I \implies \boldsymbol{B} = \frac{\mu_0 I}{2\pi r}\hat{\theta}.$$

5. The magnetic force is therefore

$$\frac{\boldsymbol{F}_B}{L} = \frac{1}{L}\int_0^L \boldsymbol{I} \times \boldsymbol{B}\, dz = \frac{1}{L}\int_0^L \frac{\mu_0 I}{2\pi r}\hat{\theta} \times I\hat{z} = \frac{1}{L}\frac{\mu_0 I}{2\pi r}(-\hat{r})\int_0^L dz = \frac{\mu_0 \lambda^2 v^2}{2\pi r}\left(-\hat{r}\right),$$

so at $r = d$, the magnetic force between the wires is attractive and

$$\frac{F_B}{L} = \frac{\mu_0 \lambda^2 v^2}{2\pi d}.$$

6. The forces are equal in magnitude when

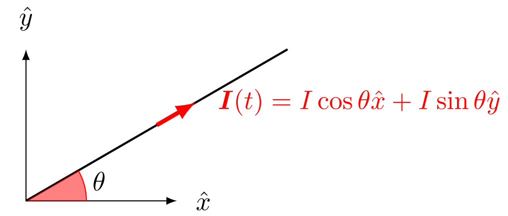
$$\frac{\lambda^2}{2\pi\varepsilon_0 d} = \frac{F}{L} = \frac{|F_B|}{L} = \frac{\mu_0\lambda^2 v^2}{2\pi r} \implies v^2 = \frac{1}{\varepsilon_0 \mu_0}.$$

One can check that $v = \sqrt{\frac{1}{\varepsilon_0\mu_0}} = c$, the speed of light.

This problem suggests that there is something strange that involves magnetism and the speed of light. Indeed, special relativity is the *cause* the magnetic force. Combining electricity and magnetism through special relativity is one of the most important parts of advanced electromagnetism courses. There's also an extremely good 5 minute YouTube video on the subject, which I recommend highly: `https://www.youtube.com/watch?v=1TKSfAkWWN0`.

Solution 16.7: See the discussion of the Hall effect in section 27.8 of Giancoli.

Solution 16.8: This is a nice problem because it involves a rare non-uniform magnetic field. Unfortunately, this makes it pretty impossible to draw a picture, so we have to fall back on being very careful with vectors. The situation is something like this.



Let us parametrize the wire by $t \in [0, L]$ where $t = 0$ is the origin and $t = L$ is the end of the wire. Then let $\boldsymbol{r}(t) = t\cos\theta\hat{x} + t\sin\theta\hat{y}$. So $\boldsymbol{I}(\boldsymbol{r}(t)) = I\cos\theta\hat{x} + I\sin\theta\hat{y}$. The force at a point $\boldsymbol{r}(t)$ is then

$$\begin{aligned}
d\boldsymbol{F}(t) &= \boldsymbol{I}(\boldsymbol{r}(t)) \times \boldsymbol{B}(\boldsymbol{r}(t))\,dt \\
&= I\left(\cos\theta\hat{x} + \sin\theta\hat{y}\right) \times B_0\left(x(t)\hat{t} + z(t)\hat{z}\right)dt \\
&= I\left(\cos\theta\hat{x} + \sin\theta\hat{y}\right) \times B_0\left(t\cos\theta\hat{x} + 0\hat{z}\right)dt \\
&= IB_0 t\sin\theta\cos\theta\ \left(\hat{y} \times \hat{x}\right)dt \\
&= -IB_0 t\sin\theta\cos\theta\hat{z}\ dt
\end{aligned}$$

Therefore

$$\boldsymbol{F} = \int_0^L d\boldsymbol{F}(t) = \int_0^L -IB_0 t\sin\theta\cos\theta\hat{z}\ dt = -IB_0\sin\theta\cos\theta\int_0^L t\,dt\hat{z} = -\frac{IB_0\sin\theta\cos\theta L^2}{2}\hat{z}.$$

Solution 16.9: See the discussion of the Hall effect in the textbook.

17 Current, Ohm's Law, & Resistance

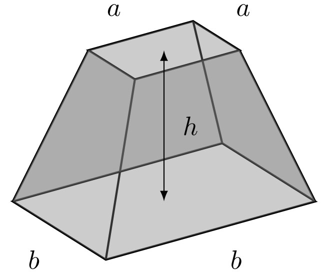
17.1 Problems

Problem 17.1: A resistor is made from two concentric spheres with radii a and b between which there is a material with resistivity that varies with radius as $\rho(r) = \rho_0\left(\frac{r}{a}\right)^s$, where ρ_0 is a constant. A battery with voltage V is hooked up to the resistor such that current is injected into the resistor from the inner sphere, and removed from the outer sphere.

1. If the electric field between the spheres is constant, what is s?

2. What is the current I in the circuit?

Problem 17.2: There is a truncated square pyramidal resistor with dimensions as shown in the figure above. If the material is made of material with resistivity ρ, what is its resistance? (If it bothers you that the current may not distribute itself evenly, you can assume $(b - a) \ll h$).

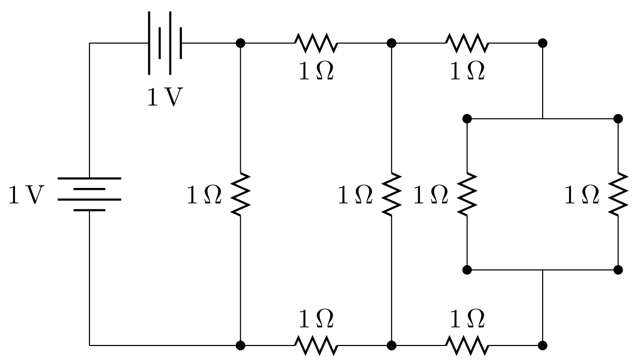


Problem 17.3: A resistor is made out of a material with resistivity ρ. The resistor is the shape of a thick cylindrical sheel of inner radius a, outer radius b, and length ℓ. The resistor is attached to the circuit at the ends of the cylinder so that the current flows over length ℓ. The following questions should all be quick after part (b).

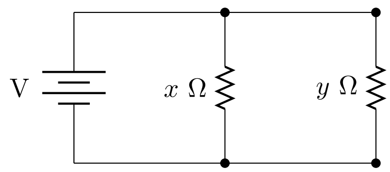
1. Draw a picture of the situation.
2. What is the resistance R of this resistor?
3. Suppose $\ell \to 2\ell$. How the R change?
4. Suppose $b \to c$ for some $c > b$. How does R change? Show that this is the same as having two resistors in parallel, one with radii a and b and the other with radii b and c.
5. If we hook the resistor up to a battery with potential different V, then what current runs through the resistor?
6. How much power is dissipated in the resistor in this situation?
7. (Optional Challenge) Instead of running the current along the length of the resistor, suppose we ran it radially from a to b. What is the resistance then?

Modified from workbook page 83.

Problem 17.4: Find the equivalent resistance for the following circuit.



Problem 17.5: Suppose we have the following circuit.



Let's find the power dissipated in two different ways and see if they're the same.

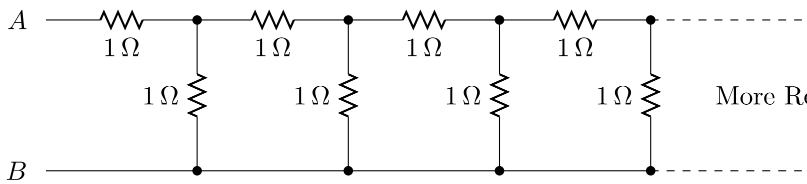
1. Find the current in each resistor and find the total power generated by summing the power generated for each resistor.
2. Find the equivalent resistance for the circuit. Then find the current through an equivalent resistor, and use this to find the power dissipated. Is this the same or not?
3. What about two resistors in series?

Problem 17.6: For some applications it is important that the value of resistance does not change. For example, suppose you want to make a resistor of resistance R and you have available two wires with resistivity ρ_1 and ρ_2, both with cross sectional area A. We want to make a resistor by combining segments of these two wires. Find the length of each wire so that the net resistance is independent of temperature. Suppose that they have fractional changes in resistance of $\alpha_1 > 0$, $\alpha_2 < 0$. Also suppose the lengths of the wires does not change with temperatures. (You may combine them in either series or parallel).

Source: Modified from Giancolli 25-26

Problem 17.7: Challenge Problem! The solution is very short, but very hard to come up with. This is more of a brain-teaser rather than a physics problem.

Find the equivalent resistance between points A and B in the infinite resistor chain below.



17.2 Solutions

Solution 17.1:

1. The *microscopic* version of Ohm's law says that $\rho \boldsymbol{J} = \boldsymbol{E}$. In this case, the current will move uniformly outwards, so $\rho(r)J(r) = E(r)$.

   The important thing to note here is that the current *density* chages with radius: if one million electrons start on the inner sphere each second, then one million must leave the outer sphere each second. However, the outer sphere is larger, so those one million electrons must be further apart from each other, so the current density much decrease as a function of r. Let I be the current in the system. Then the current density at a radius r is

   $$J(r) = \frac{I}{\text{area at radius } r} = \frac{I}{4\pi r^2}.$$

   So

   $$E(r) = \frac{I}{4\pi r^2}\rho_0 \left(\frac{r}{a}\right)^s = \frac{\rho_0 I}{a^s} r^{s-2}.$$

   The only way this can be a constant is if $s = 2$.

2. The *macroscopic* version of Ohm's law says $V = IR$. To find I, we need to know the total resistance in the circuit. The equation

   $$R = \rho \frac{L}{A}$$

   tells us how the microscopic resistivity and the macroscopic resistance are related. In this situation, the cross-sectional area A is changing as a function of radius, so we integrate:

   $$R = \int_a^b \rho(r) \frac{dr}{A(r)} = \int_a^b \rho_0 \left(\frac{r}{a}\right)^2 \frac{dr}{4\pi r^2} = \frac{\rho_0}{a^2}\int_a^b dr = \frac{\rho_0}{\pi a^2}(b-a).$$

   Therefore

   $$I = \frac{V}{R} = \frac{V\pi a^2}{\rho_0 (b-a)}.$$

Solution 17.2: We can find the total resistance by chopping the pyrimid up into slices of height dz, and integrating up their contributions from $z = 0$ to h (this is valid when $(b-a) \ll h$)

$$R = \int_0^h \frac{\rho dz}{A(z)},$$

where $A(z)$ is the width of the pyrimid at height z. Some geometry reveals that the length of one of the sides at height z is $\ell(z) = b + \frac{a-b}{h} z =$. One can check that $\ell(z = 0) = b$, $\ell(z = h) = a$, and the equation is linear. Therefore

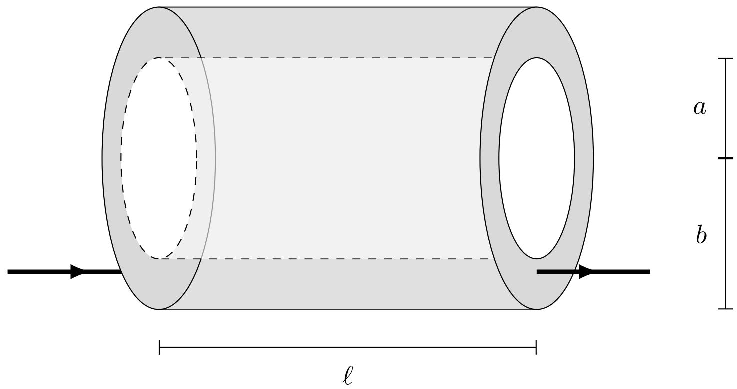
$$A(z) = \ell(z) \times \ell(z) = \left(b + \frac{a-b}{h} z\right)^2.$$

Hence

$$R = \int_0^h \frac{\rho dz}{\left(b + \frac{a-b}{h} z\right)^2} = \rho \frac{h}{a-b} \ln\left[bh + (a-b)z\right]\Big|_{z=0}^{h} = \rho \frac{h}{a-b} \ln \frac{a}{b}.$$

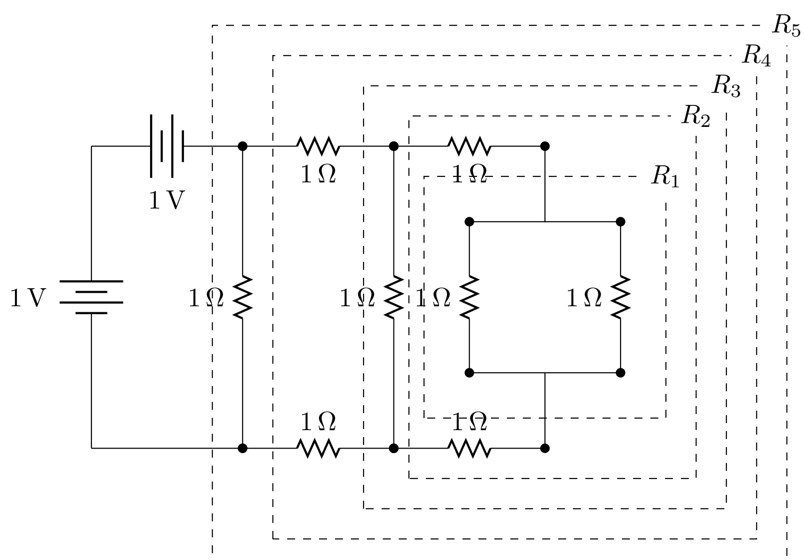
Solution 17.3:

1. Sketch.



2. Resistance is $R = \rho\frac{\ell}{A}$. In this case, the cross-sectional area is $A = \pi b^2 - \pi a^2$, so
$$R = \frac{\rho\pi\ell}{b^2 - a^2}$$
3. If $\ell \to 2\ell$, then $R \to 2R$.
4. If $b \to c > b$, then
$$R \to \frac{\rho\pi\ell}{c^2 - a^2} = R_{ca}$$
Note that
$$\frac{1}{R_{bc}} + \frac{1}{R_{ba}} = \frac{c^2 - b^2}{\pi\rho\ell} + \frac{b^2 - a^2}{\pi\rho\ell} = \frac{c^2 - a^2}{\pi\rho\ell} = \frac{1}{R_{ca}},$$
so R_{ca} is made of R_{ba} and R_{cb} in parallel.
5. By Ohm's law,
$$I = \frac{V}{R} = \frac{V(b^2 - a^2)}{\rho\pi\ell}.$$
6. Power dissipated is given by
$$P = IV = \frac{V^2(b^2 - a^2)}{\rho\pi\ell}.$$

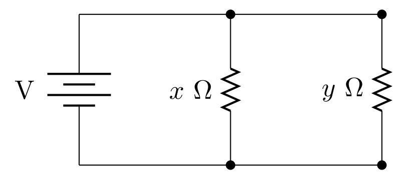
Solution 17.4: The best way to do this is probably to combine successively more resistors into equivalents starting from the right.



We then have (see picture for labelling):

$$\begin{aligned}
R_1 &= \frac{1}{\frac{1}{1}+\frac{1}{1}} = \frac{1}{2} \\
R_2 &= 1+1+\frac{1}{2} = \frac{5}{2} \\
R_3 &= \frac{1}{\frac{1}{5/2}+\frac{1}{1}} = \frac{1}{\frac{7}{5}} = \frac{5}{7} \\
R_4 &= 1+1+\frac{5}{7} = \frac{19}{7} \\
R_5 &= \frac{1}{\frac{1}{1}+\frac{7}{19}} = \frac{1}{\frac{7+19}{19}} = \frac{19}{26}\Omega.
\end{aligned}$$

Solution 17.5:



1. From Ohm's law, the current through resistor x is $I_x = \frac{V}{x}$ and through resistor y is V/y. Therefore the power generated is
$$P_x = I_x V = I_x^2 x = \frac{V^2}{x}; \quad P_y = \frac{V^2}{y}.$$
2. The total resistance is $R = \frac{1}{\frac{1}{x}+\frac{1}{y}} = \frac{1}{\frac{y+x}{xy}} = \frac{xy}{x+y}$. Therefore the current drawn from the battery is
$$I = \frac{V}{R} = \frac{V(x+y)}{xy}.$$
The power of an equivalent resistor would then be
$$P = IV = V^2\frac{(x+y)}{xy} = V^2\left(\frac{1}{x}+\frac{1}{y}\right) = P_x + P_y.$$
3. This also works for two resistors in series! In that case, we would have a current $I = V/(x+y)$ for the whole circuit, so
$$P_x = I_x^2 x = \frac{V^2 x}{(x+y)^2} \text{ and } P_y = I_y^2 y = \frac{V^2 y}{(x+y)^2}$$
But for the whole thing,
$$P = IV = \frac{V^2}{(x+y)} = \frac{V^2(x+y)}{(x+y)^2} = P_x + P_y.$$

Solution 17.6: The resistances of the two wires are therefore

$$R_1(T) = \rho_1\left(1+\alpha_1 T\right)\frac{\ell_1}{A} \text{ and } R_2(T) = \rho_2\left(1+\alpha_2 T\right)\frac{\ell_2}{A}.$$

Arrange the two resistors in series, end-to-end. We want $R := R_1(T) + R_2(T)$ to temperature-independent, i.e. $dR/dT = 0$. So

$$0 = \frac{dR}{dT} = \frac{dR_1}{dT} + \frac{dR_2}{dT} = \rho_1\alpha_1\frac{\ell_1}{A} + \rho_2\alpha_2\frac{\ell_2}{A}$$

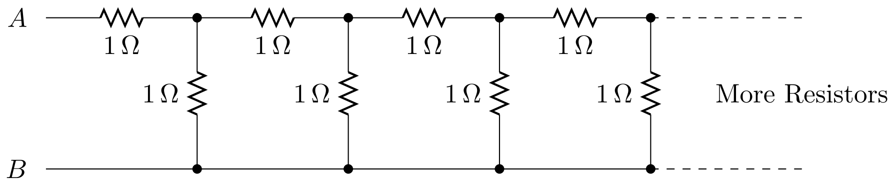
This condition is satisfied when

$$\frac{\ell_1}{\ell_2} = -\frac{\rho_2\alpha_2}{\rho_1\alpha_1}.$$

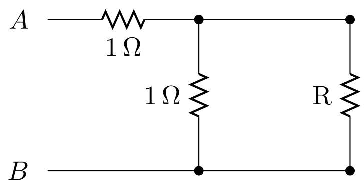
Note that $\alpha_1 > 0$ and $\alpha_2 < 0$ in the statement of the problem, so this condition on the lengths is physical.

Problem 17.8: Challenge Problem! The solution is very short, but very hard to come up with. This is more of a brain-teaser rather than a physics problem.

Notice that the circuit



is equivalent to



where the resistor R is the equivalent resistance of the entire circuit. Since this circuit is infinitely long, it has the funny property that adding one more segment to the left doesn't change anything. We therefore have the following recusion relation:

$$R = 1 + \frac{1}{\frac{1}{1} + \frac{1}{R}} = 1 + \frac{1}{\frac{R+1}{R}} = 1 + \frac{R}{R+1}$$

Rearranging,

$$R(R+1) = R + 1 + R = 2R + 1 \implies R^2 - R - 1 = 0.$$

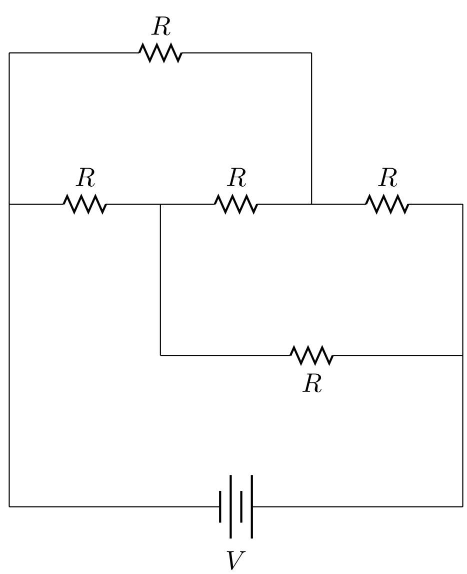
Using the quadratic formula gives $R = \frac{1+\sqrt{5}}{2}$, the Golden Ratio! I think this is an incredibly cool result. (There is a second solution to the quadratic, but it is negative and therefore non-physical.)

See also: `https://xkcd.com/356/`.

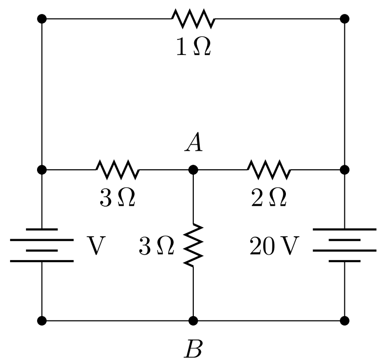
18 DC Circuits

18.1 Problems

Problem 18.1: Find the equivalent resistance of this circuit. Note that it cannot be decomposed into series and parallel.



Problem 18.2: Consider the following circuit.



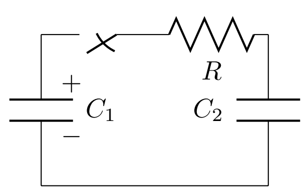
Suppose the current from A to B is 3 A.

1. Find the current through each resistor.
2. Find the voltage V of the battery on the left.
3. Find the power generated by each battery.

Source: `http://physics.info/kirchhoff/practice.shtml`.

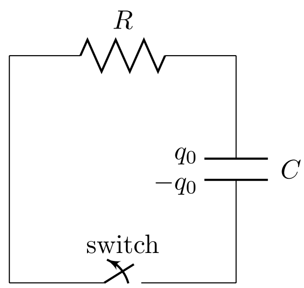
Problem 18.3: Consider the circuit shown below that contains capacitors C_1 and C_2 and resistor R. Initially the switch is open and the capacitor C_1 has charge Q_0. The switch is closed at $t = 0$.

1. What is the initial current that flows through the resistor right after the switch is closed? (4 pts)
2. After a very long time, what are the charges Q_1 and Q_2 on C_1 and C_2 respectively? (4 pts)
3. What is the charge $Q_2(t)$ on C_2 as a function of time? (5pts)
4. What is the current flowing into C_2 as a function of time? (5pts)
5. How will $Q_2(t)$ be modified if a dielectric of constant ϵ is inserted in between the plates of capacitor C_2? (3pts)



Source: Lanzara's Midterm 2, Fall 2013

Problem 18.4: In the RC circuit shown below the capacitors caries an initial charge q_0. The switch is opened at time $t = 0$.



1. What will happen when the switch is closed?
2. Immediately after the switch is closed, how much current flows in the circuit?

3. After a long time, how much current flows in the circuit? Why? What is the voltage across the capacitor at this point?
4. Sketch graphs showing the charge on the capacitor and current flowing in the circuit as functions of time.
5. How much energy was stored in the electric field of the capacitor initially?
6. How much energy is stored in the electric field of the capacitor after a long time?
7. What happened to this energy?

Source: 7B workbook, Problem 2, page 126.

18.2 Solutions

Solution 18.1: One may apply Kirchoff's laws to solve this.

Solution 18.2: See `http://physics.info/kirchhoff/practice.shtml`. This is actually a pretty good website and has several other good practice problems — with solutions!

Solution 18.3: See the solution to Lanzara's Midterm 2, Fall 2013, problem 4 at `https://tbp.berkeley.edu/exams/4198/download/`.

Solution 18.4:

1. When the switch is closed, the charge will run "downhill" the potential gradient and produce a counterclockwise current in the circuit.
2. Immediately after the switch is closed, Ohm's law says
$$I(t=0) = \frac{V}{R} = \frac{q_0/C}{R}.$$
3. After a long time, charge q_0 has flowed around the whole circuit and cancelled out the negative charge on the other plate. Nothing will flow after this and the voltage across the capacitor will be zero, because there is no net charge on either plate.
4. By Kirchoff's law, as we go around the circuit counterclockwise,
$$0 = V_{\text{capacitor}} + V_{\text{resistor}} = \frac{Q}{C} + IR = \frac{1}{C}Q(t) + \frac{dQ}{dt}R \implies \frac{dQ}{dt} = \frac{1}{RC}Q(t).$$
This is a differential eqution subject to the initial condition $Q(t=0) = q_0$. One may solve it by dividing through by $Q(t)$ and then integrating both sides with respect to t:
$$\frac{1}{Q}\frac{dQ}{dt} = -\frac{1}{RC} \implies \frac{dQ}{Q} = \frac{dt}{RC} \implies \int \frac{dQ}{Q} = -\int \frac{dt}{RC} \implies \ln Q = -\frac{t}{RC} + \text{const}.$$
We have a constant because this was an indefinite integral. Taking the exponential of both sides,
$$Q(t) = \exp\left(\frac{-t}{RC} + C\right) = Ae^{t/RC}$$
where I have renamed $A = e^{\text{const}}$. Plugging in the initial condition $Q(t=0) = q_0$ shows $A = q_0$. Therefore
$$Q(t) = q_0 e^{-t/RC}.$$
The current flowing is
$$I = \frac{dQ}{dt} = -\frac{q_0}{RC}e^{-t/RC}.$$
5. The initial energy stored was $U = \frac{1}{2}CV^2 = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}\frac{q_0^2}{C}$.
6. After a long time, there is no potential difference, so $U = \frac{1}{2}CV^2 = 0$.
7. This energy was dissipated in the resistor; one can check that the power generated by the resistor, $P(t) = I(t)V(t)$, integrated from $t = 0$ to ∞ matches the initial amount of energy in the capacitor.

19 Ampère's Law

Last week we focused on finding the magnetic force produced when we knew the magnetic field. This week we will go about finding the magnetic force produced by a series of wires or other currents. There are two main ways to do this:

1. Ampère's Law. This can be thought of as "Gauss's Law for magnetism" — it's basically the same as Gauss's law but down one dimension. Like Gauss's law, it needs high symmetry to be useful.
2. The Biot-Savart law. This is the analogue of finding the electric field through superposition of point charges. In this case, however, we can this of using "superposition of small segments of wires". Due to the cross-product inside the integram, applying this is often very messy and should be avoided where possible.

19.1 Helpful Equations

$$\boldsymbol{F} = q\boldsymbol{v} \times \boldsymbol{B}. \qquad \text{(Coulomb's Law)}$$

$$d\boldsymbol{F} = I d\hat{\ell} \times \boldsymbol{B} \qquad \text{(Infinitesimal Coulomb's Law for a Wire)}$$

$$\oint_{\text{loop}} \boldsymbol{B} \cdot d\hat{\ell} = \mu_0 I_{\text{enc}} \qquad \text{(Ampère's Law)}$$

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \int_{\text{current}} \frac{\boldsymbol{J} \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} = \frac{\mu_0}{4\pi} \int_{\text{wire}} \frac{I d\hat{\ell} \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \qquad \text{(Biot-Savart Law)}$$

19.2 Problems

Problem 19.1: A thick wire of radius R carries total current I distributed uniformly across it.

1. Sketch the magnetic field created by the wire at points both inside and outside the wire.
2. Find the current density $\boldsymbol{K}$ inside the wire. What are the units of this quantity?
3. Find the magnetic field (both magnitude *and* direction) at all points in space.

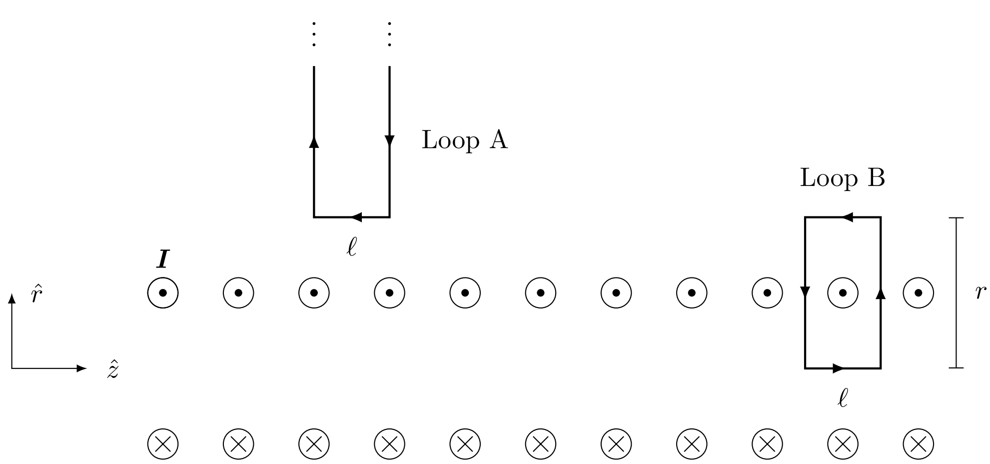
Source: 7B workbook.

Problem 19.2: A thick wire of radius R carries total current I distributed *non-uniformly*. The current density within the wire is $\boldsymbol{K}(r) = K_0 \frac{r^4}{R^4} \hat{z}$.

1. Sketch the magnetic field both inside and outside the wire.
2. Find the magnetic field (both magnitude *and* direction) at all points in space.

Source: 7B workbook.

Problem 19.3: An infinitely long solenoid with n turns per unit length carries a current I. This problem will walk through how to calculate its magnetic field. The figure shows a side view.



1. Set up a cylindrical coordinate system (r, θ, z). Keeping in mind that the solenoid is infinitely long, which of the three coordinates will the magnetic field strength depend on?

2. Sketch the *direction* of the magnetic field both inside and outside the solenoid. (That is to say, wave your right hand around a lot until you know which direction the vectors should go.)

3. Write the most general form of the magnetic field in this setup. (This is something similar to the expressions like $\boldsymbol{E} = E(z)\hat{z}$ that we used with Gauss's law.)

4. Consider Loop A. This ... mean that the line extends to infinity. Using the fact that the magnetic field must decay to zero infinitely far away, find the value of

$$\oint \boldsymbol{B} \cdot d\boldsymbol{\ell}$$

for Loop A. What is the current enclosed by the loop? Deduce what the $\boldsymbol{B}$ field is outside the solenoid.

5. Apply Ampère's Law to loop B to find the magnetic field inside the solenoid. Assume that the wires that make up the solenoid have zero width.

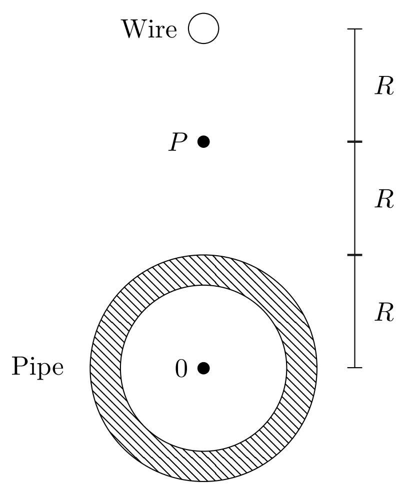
6. Write down the magnetic field at all points in space.

Source: 7B workbook.

Problem 19.4: An infinite slab extending from $z = -a$ to $z = a$ carries a uniform volume current $\vec{J} = J\hat{x}$. Find the magnetic field, as a function of z, both inside and outside the slab.

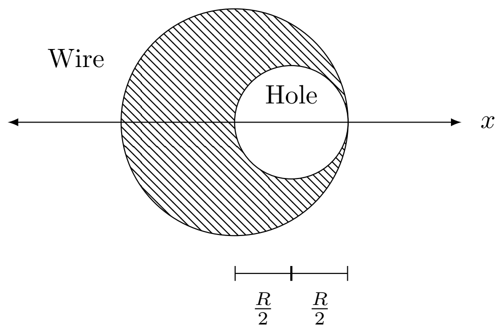
Source: Griffiths 5.14

Problem 19.5: In the figure below, a long circular pipe with outside radius R carries a (uniformly distributed) current I into the page. A wire runs parallel to the pipe at a distance of $3R$ from center to center. Find the magnitude and direction (into or out of the page) of the current in the wire such that the net magnetic field at point P has the same magnitude as the net magnetic field at the center of the pipe but is in the opposite direction.



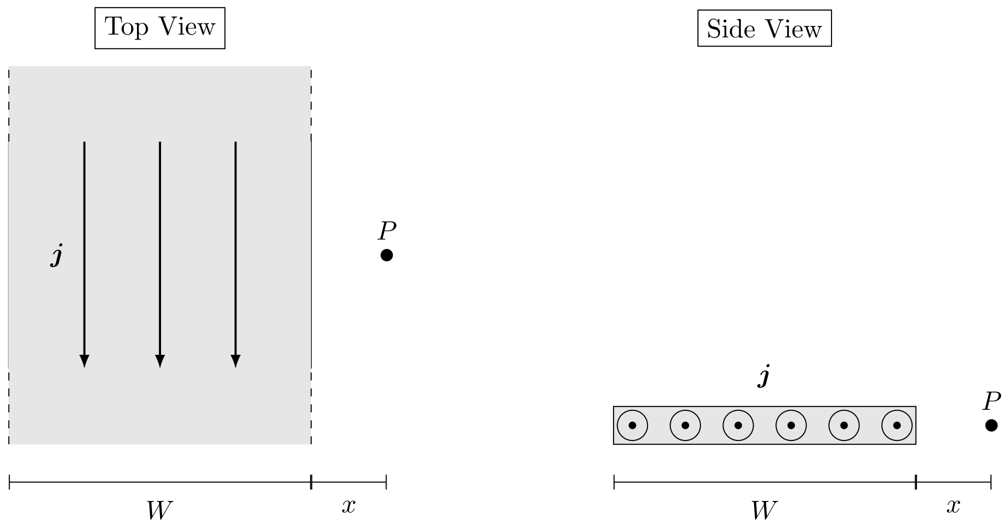
Source: Halliday and Resnick 29-48.

Problem 19.6: There is a long wire with radius R. A hole of radius $\frac{R}{2}$ is cut out of the wire, centered at a position $\frac{R}{2}$ away from the center. If a current density per unit area $\boldsymbol{J}$ flows through the wire with the hole cut out, what is the magnetic field on the x-axis? (See Picture.)



Source: Physics 7B Workbook Supplement, 10.2.2

Problem 19.7: The figure below shows a thin sheet of current with a current *per unit width* $\boldsymbol{j}$ that is infinitely long, but has a width W . Find the magnetic field (direction and magnitude) at the point P.



Source: Speliotopoulos Final Fall 2012

Problem 19.8: Consider a long cable made of two coaxial thin cylinders. The inner cylinder had radius r_1 and the outer cylinder has radius r_2. The inner cylinder carries a current $+I$ and $-I$ flows in the outer conductor.

1. Compute the magnetic field B for the three regions: $r < r_1$; $r_1 < r < r_2$; and $r > r_2$ Neglect the thickness of the conductors. Do not just write down the answer from your notes. Show the calculation
2. Use the result in part a) to compute the magnetic energy stored (per unit length) in the cable.

Problem 19.9: A solenoid produces a uniform magnetic field B in the x-axis. Suppose a charged particle q moves with velocity (v_x, v_y, v_z) within the solenoid.

1. Find the magnetic force.
2. Describe the motion of the particle. Draw its trajectory.
3. If the solenoid has finite length, the particle will eventually escape the magnetic field. However, if we bend the solenoid around and connect the ends to each other, the particle will be trapped in a circulation. This configuration of coils is called a toroid. It is so useful in plasma physics that physicists decide to give it a cool Russian name called a *Tokamak*. If we want to create a magnetic field of field strength B_0 in a Tokamak of major radius R and minor radius r (that is to say, the solenoid is of radius r), given a current of I, what is the number of coils per unit length we will need?
4. What is the magnetic energy stored inside the torus, given the above? (*Hint:* $u = \frac{1}{2\mu_0}B^2$)

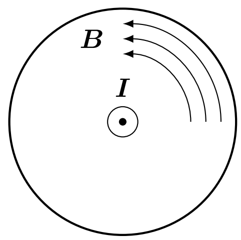
5. The problem with using Tokamaks to confine plasma is that the charged particles tend to "leak" out after some time. What do you think is the problem? Is the field in the toroid truly uniform?

6. What are some ways to remedy this problem? (Of interest: stellarators e.g. the newest Wendelstein 7-X)

19.3 Solutions

Solution 19.1:

1. The magnetic field goes in the $\hat{\theta}$ direction.

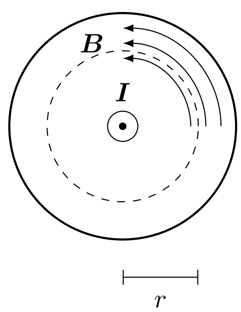


2. The current density is

$$K = \frac{\text{current}}{\text{area}} = \frac{I}{\pi R^2}$$

with units of Coulombs per unit time per unit area.

3. We can use Ampère's law. By symmetry, the magnetic field is of the form $\boldsymbol{B} = B(r)\hat{\theta}$. Consider an Amperian loop of radius r (dashed line in the picture).



Then the line integral gives

$$\int_{\text{loop}} \boldsymbol{B} \cdot d\boldsymbol{\ell} = \int_0^{2\pi} B(r)\hat{\theta} \cdot \hat{\theta}\, r d\theta = B(r) 2\pi r.$$

Meanwhile the charge enclosed is

$$I_{\text{enc}}(r) = \int K\, dA = \int_0^r K 2\pi r\, dr = K\pi r^2 = \frac{Ir^2}{R^2} \text{ for } r < R$$

and $I_{\text{enc}}(r) = I\pi R^2$ for $r > R$. Therefore

$$\boldsymbol{B} = \frac{\mu_0}{2\pi r} I_{\text{enc}}\hat{\theta} = \frac{\mu_0 I}{2\pi} \begin{cases} \frac{r}{R} & r < R \\ \frac{R^2}{r} & r > R. \end{cases} \hat{\theta}$$

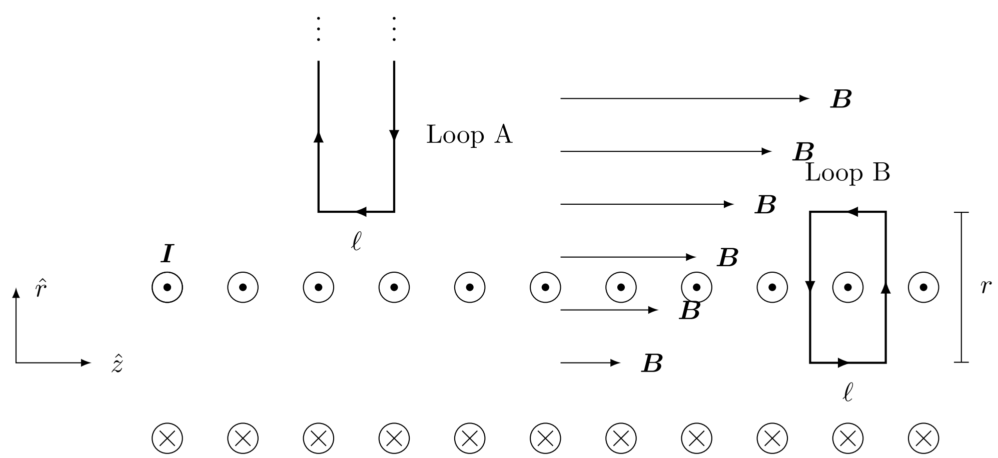
Solution 19.2:

1. This is the same as above except that the magnetic field will now have a different radial dependence.

2. The only thing that changes from above is the enclosed current. Inside the wire this is
$$I_{\text{enc}}(r) = \int K(r)\,dA = \int_0^r K_0 \frac{r^4}{R^4} 2\pi r\,dr = \frac{2\pi K_0}{R^4}\frac{r^5}{5}.$$
But outside the wire
$$I_{\text{enc}}(r) = \int K(r)\,dA = \int_0^R K_0 \frac{r^4}{R^4} 2\pi r\,dr = \frac{2}{5}\pi K_0 R.$$
Therefore
$$\boldsymbol{B}(r) = \frac{\mu_0}{2\pi r} I_{\text{enc}}\hat{\theta} = \frac{2\pi K_0}{5}\frac{1}{2\pi r}\begin{cases}\frac{r^5}{R^4} & r < R \\ R & r < R\end{cases}\hat{\theta} = \frac{K_0}{5}\begin{cases}\frac{r^4}{R^4} & r < R \\ \frac{R}{r} & r < R\end{cases}\hat{\theta}.$$

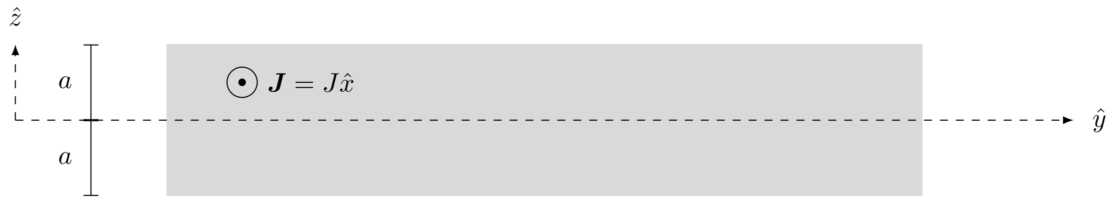
Solution 19.3:

1. The form of the magnetic field will only depend on the radial coordinate; we have symmetry in all other directions.
2. The *direction* field will look like this. Of course, we don't know what the radial dependence is yet, so I've drawn something fairly arbitrary.



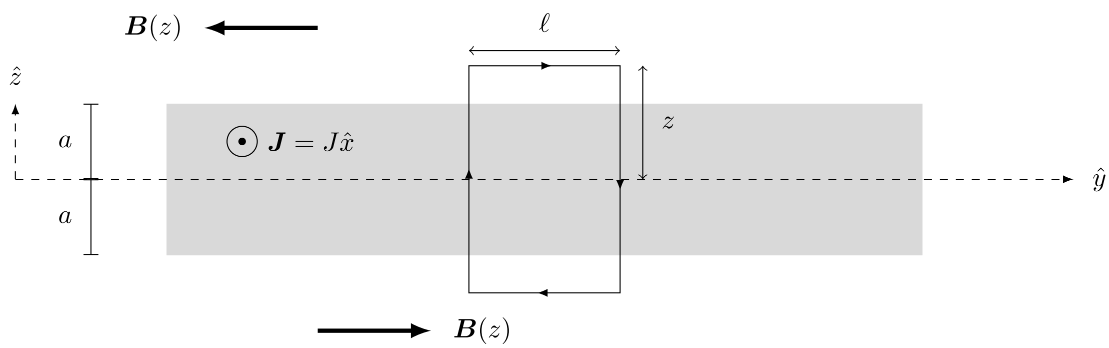
3. The magnetic field will look like $\boldsymbol{B} = B(r)\hat{z}$.
4. Loop A is intended to extend to infinity. Because of the form of $\boldsymbol{B}$, we know that only the sides of length ℓ will contribute. Therefore
$$\int_\ell^0 \boldsymbol{B}\cdot d\hat{z} + 0 = -B(r)\ell + 0 = \mu_0 I_{\text{enc}} = 0$$
since there is no enclosed charge. Therefore $B(r) = 0$ outside the solenoid.
5. For loop B, the vertical lines in the picture will not contribute because they're perpendicular to the magnetic field and the side of length ℓ outside will not contribute either because we just found $\boldsymbol{B} = 0$ there. So we're left with
$$\oint \boldsymbol{B}\cdot d\hat{z} = B(r)\ell = \mu_0 I_{\text{enc}}.$$
We've enclosed on of the sides for a length ℓ, so
$$I_{\text{enc}} = \frac{\text{turns}}{\text{length}} \times \text{length} \times \text{current in the wire} = n\ell I.$$
Thus
$$B(r)\ell = \mu_0 n\ell I \implies B(r) = nI.$$
6. Combining everything we have learned, the magnetic field is uniform and constant inside the solenoid and vanishes outside:
$$\boldsymbol{B} = \hat{z}\begin{cases}\mu_0 nI & r < R \\ 0 & r > R\end{cases}$$

Solution 19.4: Let's draw a picture to start with. I can't actually draw an infinitely wide slab, so I'll draw one of finite width and we can pretend it goes to infinity.



In this situation, we have translational symmetry in the xy plane and we can flip about the z-axis. From the symmetry, we know that the magnetic field will have the form $\boldsymbol{B} = B(z)\hat{y}$ where $B(z) = -B(-z)$. That is, the magnetic field will point left above the z-axis and point right below the z-axis.

To find $B(r)$, we should consider an Amperian loop that respects the symmetry.



The integral over the Amperian loop therefore picks up equal contributions from the top and bottom of the loop and nothing from the sides since $\boldsymbol{B} \perp d\boldsymbol{\ell}$ there:

$$\oint \boldsymbol{B} \cdot d\boldsymbol{\ell} = 2B(z)\ell.$$

We also want to know the charge enclosed. For $z < a$, the loop has interior area $2z\ell$ so $I_{\text{enc}}(z) = 2z\ell J$. For $z > a$, there is only current flowing through an area $2a\ell$ inside the loop, so $I_{\text{enc}}(z) = 2a\ell$ in this case. By Ampère's Law,

$$\boldsymbol{B}(z) = \frac{\mu_0 I_{\text{enc}}}{2\ell}\hat{y} = 2\mu_0 J \begin{cases} z & 0 \le z < a \\ a & z > a \end{cases} \hat{y},$$

and for $z < 0$ we know the field from $B(z) = -B(-z)$.

Solution 19.5: External to the wire the magnetic field of the pipe is indistinguishable from that of a wire, so

$$\boldsymbol{B}_{\text{pipe}}(r > R) = -\frac{\mu_0 I}{2\pi r}\hat{\theta}.$$

The minus sign is because the current is into the page. So in particular $\boldsymbol{B}(P) = \frac{\mu_0 I}{2\pi(2R)}\hat{x}$ where I have taken $\hat{x}$ to point to the right in the picture. At the center of the pipe, which I will call 0, there is no magnetic field from the pipe by Ampère's Law.

Meanwhile the field from the wire at P is

$$\boldsymbol{B}_{\text{wire}}(P) = \frac{\mu_0 I_w}{2\pi R}\hat{x}$$

and at the origin is

$$\boldsymbol{B}_{\text{wire}}(O) = \frac{\mu_0 I_w}{2\pi(3R)}\hat{x}.$$

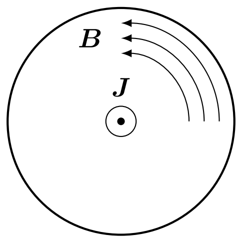
Here I have taken I_w to be the (possibly negative) strength of the current coming out of the page in the wire. We want to satisfy the condition $\boldsymbol{B}(P) = -\boldsymbol{B}(0)$. So:

$$\boldsymbol{B}(P) = \boldsymbol{B}_{\text{pipe}}(P) + \boldsymbol{B}_{\text{wire}}(P) = \frac{\mu_0 I}{2\pi(2R)}\hat{x} + \frac{\mu_0 I_w}{2\pi R}\hat{x}$$

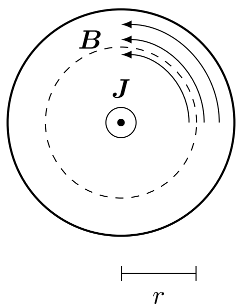
and
$$\boldsymbol{B}(0) = \boldsymbol{B}_{\text{wire}}(0) = \frac{\mu_0 I_w}{2\pi(3R)}\hat{x}.$$
So our condition means that
$$\frac{\mu_0 I}{2\pi(2R)} + \frac{\mu_0 I_w}{2\pi R} = -\frac{\mu_0 I_w}{2\pi(3R}.$$
Dividing both sides by $\frac{\mu_0}{2\pi R}$,
$$\frac{I}{2} + I_w = -\frac{I_w}{3} \implies I_w = -\frac{3}{8}I.$$

Solution 19.6: We can think of this as the superposition of a wire of radius R with current $\boldsymbol{J}$ and a wire of radius $R/2$ with current $-\boldsymbol{J}$. For each shape, the magnetic field can be found via Ampere's law, and then we can apply superposition.

Let's consider the wire of radius R first. The magnetic field goes in the $\hat{\theta}$ direction and the current density is
$$K = \frac{\text{current}}{\text{area}} = \frac{I}{\pi R^2}$$
with units of Coulombs per unit time per unit area.



By symmetry, the magnetic field is of the form $\boldsymbol{B} = B(r)\hat{\theta}$. Consider an Amperian loop of radius r (dashed line in the picture).



Then the line integral gives
$$\int_{\text{loop}} \boldsymbol{B}\cdot d\boldsymbol{\ell} = \int_0^{2\pi} B(r)\hat{\theta}\cdot\hat{\theta}\, r d\theta = B(r)2\pi r.$$
Meanwhile the charge enclosed is
$$I_{\text{enc}}(r) = \int J\, dA = J\pi r^2 \text{ for } r < R$$
and $I_{\text{enc}}(r) = J\pi R^2$ for $r > R$. Therefore
$$\boldsymbol{B} = \frac{\mu_0}{2\pi r} I_{\text{enc}}\hat{\theta} = \frac{\mu_0 J}{2\pi}\begin{cases} r & r < R \\ \frac{R^2}{r} & r > R. \end{cases}\hat{\theta}$$

Letting the $\hat{z}$ direction point up in the picture, then the magnetic field is
$$\boldsymbol{B} = \boldsymbol{B}_R + \boldsymbol{B}_{R/2} = \frac{\mu_0 J}{2\pi}\begin{cases} -\hat{z}\frac{R^2}{|x|} & x < -R \\ -\hat{z}\,|x| & -R < x < 0 \\ \hat{z}x & 0 < x < R \\ \hat{z}\frac{R^2}{x} & R < x \end{cases} - \frac{\mu_0 J}{2\pi}\begin{cases} -\hat{z}\frac{R^2}{4|x-R/2|} & x < 0 \\ -\hat{z}\,|x-R/2| & 0 < x < R/2 \\ \hat{z}x & R/2 < x < R \\ \hat{z}\frac{R^2}{4x} & R < x. \end{cases}$$

Combining these, we get

$$\boldsymbol{B}(x) = \frac{\mu_0 J}{2\pi}\begin{cases} -\hat{z}\left(\frac{R^2}{|x|} + \frac{R^2}{4|x-R/2|}\right) & x < -R \\ -\hat{z}\left(|x| + \frac{R^2}{4|x-R/2|}\right) & -R < x < 0 \\ \hat{z}\left(x - \left(\frac{R}{2} - x\right)\right) & 0 < x < \frac{R}{2} \\ \hat{z}\left(x + \left(\frac{R}{2} - x\right)\right) & \frac{R}{2} < x < R \\ \hat{z}\left(\frac{R^2}{x} + \frac{R^2}{4(x-R/2)}\right) & x > R. \end{cases}$$

This is actually terribly ugly.

Solution 19.7: We know that for a infinite wire with a current I, the magnetic field is

$$\boldsymbol{B} = \frac{\mu_0 I}{2\pi r}\hat{\theta}.$$

We can find the field at point P by integrating across the plate. If we consider a small part of the sheet of width dr, then it has current $dI = jdr$. By the right hand rule, we can see the magnetic field from each dr will point in the vertical direction, which I will call $\hat{z}$. Therefore

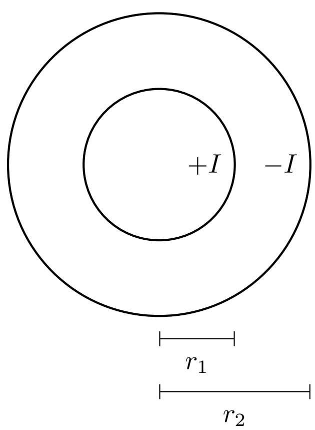
$$\boldsymbol{B} = \int_x^{x+W} \frac{\mu_0 j dr}{2\pi r}\hat{z} = \frac{\mu_0 j}{2\pi}\ln\left(1 + \frac{W}{x}\right).$$

One can also use the Biot-Savart law here, but it's much longer.

Alternatively, see `https://tbp.berkeley.edu/exams/4317/download/`. Note that the solution says "Gauss's Law" when it means "Ampère's Law".

Solution 19.8:

1. The situation is as follows.



There is no enclosed charge for $r < r_1$ or $r > r_2$, so from Ampère's Law it is easy to see that

$$\boldsymbol{B} = \begin{cases} 0 & r < r_1 \\ \frac{\mu_0 I}{2\pi r}\hat{r} & r_1 < r < r_2 \\ 0 & r > r_2. \end{cases}$$

2. The energy stored in a magnetic field at a point is $U = \frac{\mu_0}{2}|\boldsymbol{B}|^2$, so the energy in a length ℓ is

$$U = \int \frac{1}{2\mu_0}|\boldsymbol{B}|^2\, dV = \int_0^\ell \int_0^{2\pi}\int_{r_1}^{r_2} \frac{1}{2\mu_0}\left(\frac{\mu_0 I}{2\pi r}\right)^2 dr\, r d\theta\, dz = \frac{1}{2\mu_0}\left(\frac{\mu_0 I}{2\pi}\right)^2 \ell 2\pi \int_{r_1}^{r_2}\frac{r^2}{r}\,dr = \frac{\mu_0 I^2}{2}\ell \ln\frac{r_2}{r_1}.$$

So the energy stored per unit length is

$$\frac{U}{\ell} = \frac{\mu_0 I^2}{2}\ln\frac{r_2}{r_1}.$$

Solution 19.9:

1. The magnetic field of a solenoid is $\boldsymbol{B} = \mu_0 I n \hat{z}$, so for general $\boldsymbol{v} = (v_x, v_y, v_z)$,
$$\boldsymbol{F} = q\boldsymbol{v} \times \boldsymbol{B} = q\mu_0 I n \left(v_y \hat{x} - v_x \hat{y}\right).$$
2. The particle spirals down the solenoid in a helical motion.
3. One needs $n = \frac{B_0}{\mu_0 I}$ to get a magnetic field of magnitude B_0.
4. Using $U = \frac{1}{2\mu_0} |\boldsymbol{B}|^2$, one can integrate to find
$$U = \frac{\pi^2 r^2 R B^2}{\mu_0}$$
where B is the magnitude of the magnetic field.
5. The coils are denser near the center of the toroid, hence the magnetic field is not uniform, and charged particles leak out the gaps.
6. One could adjust the density of the coils such that a uniform magnetic field is maintained. (This designing technology is only beginning to open up to us because of supercomputers.)

20 Biot-Savart Law

Biot-Savart problems tend to be extremely ugly. Though the following problems are quick to state, the calculations are rather involved, requiring careful attention to the geometry. Drawing pictures will make your life much easier! In fact, I would recommend trying to figure out what direction the magnetic field will go first using the right-hand rule, and only computing the component of the $\boldsymbol{B}$-field in that direction.

Be warned that these can take rather a long time until you get some practice with them! As with electrostatics problems where you compute the electric field using superposition, there are only so many problems of this type it is possible to ask, and those below are the most common.

20.1 Helpful Equations

$$\boldsymbol{F} = q\boldsymbol{v} \times \boldsymbol{B}. \qquad \text{(Coulomb's Law)}$$

$$d\boldsymbol{F} = I d\hat{\ell} \times \boldsymbol{B} \qquad \text{(Infinitesimal Coulomb's Law for a Wire)}$$

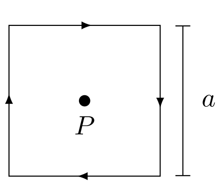
$$\oint_{\text{loop}} \boldsymbol{B} \cdot d\hat{\ell} = \mu_0 I_{\text{enc}} \qquad \text{(Ampère's Law)}$$

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \int_{\text{current}} \frac{\boldsymbol{J} \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} = \frac{\mu_0}{4\pi} \int_{\text{wire}} \frac{I d\hat{\ell} \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \qquad \text{(Biot-Savart Law)}$$

20.2 Problems

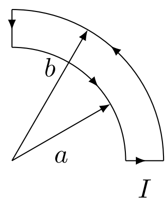
Problem 20.1: (Magnetic Field from a Finite Wire) Find the magnetic field a distance s from a straight wire of length L carrying a steady current I. This is a shockingly nasty problem that basically requires trig substitution to write down what $d\ell$ is. You may want to use the fact that $\frac{d}{d\theta} \tan\theta = \sec^2\theta$. *Source*: Griffiths EM, Example 5.5

Problem 20.2: (Magnetic Field from a Square of Wire) Suppose there is a square loop of wire with side length a. If the loop carries a steady current I, what is the magnetic field in the center?
Source: Griffiths EM, Problem 5.8.



Problem 20.3: (Magnetic Field above a Loop) Suppose there is a circular wire loop of radius R. If the loop carries a steady current I, what is the magnetic field a distance z above the center? Be careful with your geometry here! Unless you draw a very precise picture, an angle α and $\frac{\pi}{2} - \alpha$ will look the same. *Source: Griffiths EM, Example 5.6.*

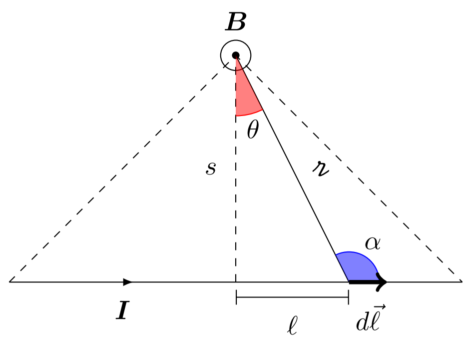
Problem 20.4: (Magnetic Field from a Quarter Circle) Suppose there is a wire in the shape shown below with a steady current I travelling clockwise. What is the magnetic field at point P?



Problem 20.5: Workbook, page 97, challenge problem: finding the magnetic field of a dipole off-axis.

20.3 Solutions

Solution 20.1:



By using the right hand rule or intuition, the direction of the magnetic field is out of the page. m. Let α and θ be as labelled in the diagram. We now want to find

$$d\boldsymbol{B} = \frac{\mu_0 I}{4\pi} \frac{d\vec{\ell} \times \hat{\boldsymbol{\mathscr{r}}}}{r^2}.$$

We already know the direction of the crossproduct is in out of the page, so we only care about the magnitude now. The magnitude of the cross-product is therefore

$$\left| d\vec{\ell} \times \hat{\boldsymbol{\mathscr{r}}} \right| = \sin\alpha d\ell = \cos\theta d\ell.$$

By the definition of cosine,

$$\cos\theta = \frac{s}{\mathscr{r}} \implies \frac{1}{\mathscr{r}^2} = \frac{\cos^2\theta}{s^2}.$$

However, to actually compute this, we want to integrate with respect to theta. From the definition of tangent,

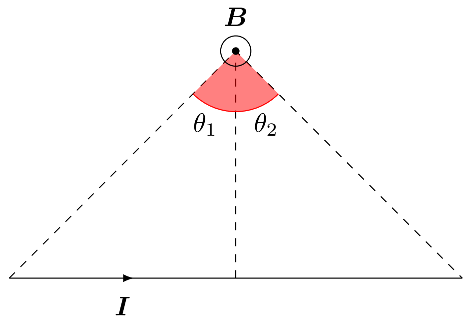
$$\tan\theta = \frac{\ell}{s} \implies d\ell = sd(tan\theta) = \frac{s}{\cos^2\theta} d\theta.$$

Putting these together,

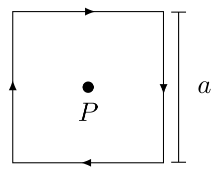
$$dB = \frac{\mu_0 I}{4\pi} \frac{\cos^2\theta}{s^2} \cos\theta \frac{s}{\cos^2\theta} d\theta = \frac{\mu_0 I}{4\pi} \frac{\cos\theta}{s} d\theta.$$

We now integrate this from θ_2 to θ_1 (see picture). Therefore

$$\boldsymbol{B} = \int_{\theta_1}^{\theta_2} d\boldsymbol{B}(\theta) = \frac{\mu_0 I}{4\pi s} \int_{\theta_1}^{\theta_2} \cos\theta d\theta = \frac{\mu_0 I}{4\pi s} (\sin\theta_2 - \sin\theta_1).$$



Solution 20.2:



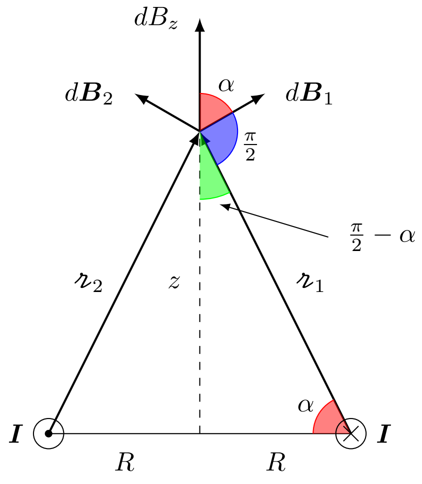
Each of the sides contributes equally to the final magnetic field, which will be out of the board. The magnetic field due to each side can be found immediately by using the first problem with $s = a/2$, and $\theta_2 = -\theta_1 = \frac{\pi}{4}$. Then:

$$\boldsymbol{B}_{\text{one side}} = \frac{\mu_0 I}{4\pi(a/2)}\left(\frac{\sqrt{2}}{2} - (-1)\frac{\sqrt{2}}{2}\right) = \frac{\mu_0 I\sqrt{2}}{2\pi a}.$$

So

$$\boldsymbol{B}_{\text{total}} = 4\boldsymbol{B}_{\text{one side}} = \frac{2\sqrt{2}\mu_0 I}{\pi a}.$$

Solution 20.3: Consider a side view of the problem.



Considering the small part of the wire on the right-hand bottom, the cross-product will give a vector $d\boldsymbol{B}_1$. However, the contribution from the other side of the circle will give $d\boldsymbol{B}_2$. These will clearly cancel out when we integrate around the circle, so we only need to keep track of the vertical component. Therefore we want to compute:

$$dB_Z = \frac{\mu_0 I}{4\pi}\frac{\left(d\hat{\ell} \times \hat{r}\right)_z}{r^2}.$$

First,

$$\left|d\hat{\ell} \times \hat{r}\right| = \sin\frac{\pi}{2} d\ell = d\ell$$

so the z-component is the projection along the z-axis:

$$\left(d\hat{\ell} \times \hat{r}\right)_z = (d\ell)_z = \cos\alpha d\ell = \frac{R}{r}d\ell.$$

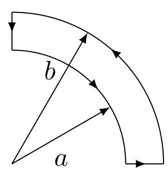
From the pythagorean theorem, $r = \sqrt{R^2 + z^2}$, so

$$\begin{aligned} dB_Z &= \frac{\mu_0 I}{4\pi} \frac{\left(d\hat{\ell} \times \hat{r}\right)_z}{r^2} \\ &= \frac{\mu_0 I}{4\pi} \frac{1}{r^2} \frac{R}{r} d\ell \\ &= \frac{\mu_0 I}{4\pi} \frac{R}{r^3} R d\theta \\ &= \frac{\mu_0 I}{4\pi} \frac{R^2}{\left(R^2 + z^2\right)^{3/2}} d\theta. \end{aligned}$$

Integrating this gives

$$B = \int_0^{2\pi} dB_z = \frac{\mu_0 I}{4\pi} \frac{R^2}{\left(R^2 + z^2\right)^{3/2}} 2\pi = \frac{\mu_0 I}{2} \frac{R^2}{\left(R^2 + z^2\right)^{3/2}}.$$

Solution 20.4:



This can also be done re-using old problems. The two straight sides of the wire are parallel or anti-parallel to the $\hat{r}$ vector, so the cross-products vanish there. They therefore do not contribute to the magnetic field. Using the right-hand rule, both semi-circles contribute a component in the vertical direction. To find the magnitudes of those components, it is easiest to find the magnetic field due to a quarter circle of arbitrary radius and current: $\boldsymbol{B}_{\text{Q.C.}}(R, I)$.

However, a quarter circle is just a quarter of a loop, so we can just use dB_z from above, with $z = 0$. Therefore

$$\boldsymbol{B}_{\text{Q.C.}}(R, I) = \int_0^{\pi/2} \frac{\mu_0 I}{4\pi} \frac{R^2}{\left(R^2 + 0^2\right)^{3/2}} d\theta \hat{z} = \frac{\mu_0 I}{8R} \hat{z}.$$

The total magnetic field is then:

$$\boldsymbol{B} = \boldsymbol{B}_{\text{Q.C.}}(a, -I) + \boldsymbol{B}_{\text{Q.C.}}(b, I) = \frac{\mu_0 I}{8} \left(\frac{1}{b} - \frac{1}{a}\right) \hat{z}$$

Solution 20.5:

21 Faraday's Law and Magnetic Induction

This week we will apply Faraday's law to circuits. The phenomena of self-inductance gives us a novel circuit component: the inductor. Just as capacitors can temporarily store electric fields, inductors can store magnetic fields.

21.1 Helpful Equations

$$\boldsymbol{F} = q\boldsymbol{v} \times \boldsymbol{B}. \qquad \text{(Coulomb's Law)}$$

$$d\boldsymbol{F} = I d\hat{\ell} \times \boldsymbol{B} \qquad \text{(Infinitesimal Coulomb's Law for a Wire)}$$

$$\oint_{\text{loop}} \boldsymbol{B} \cdot d\hat{\ell} = \mu_0 I_{\text{enc}} \qquad \text{(Ampère's Law)}$$

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \int_{\text{current}} \frac{\boldsymbol{J} \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} dV = \frac{\mu_0}{4\pi} \int_{\text{wire}} \frac{\boldsymbol{I} \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} d\ell \qquad \text{(Biot-Savart Law)}$$

$$\mathcal{E} = \oint_{\text{loop}} \boldsymbol{E} \cdot d\boldsymbol{\ell} = -\frac{d\Phi_B}{dt} \qquad \text{(Faraday's Law)}$$

$$U = \frac{1}{2} L I^2 \qquad \text{(Energy in an Inductor)}$$

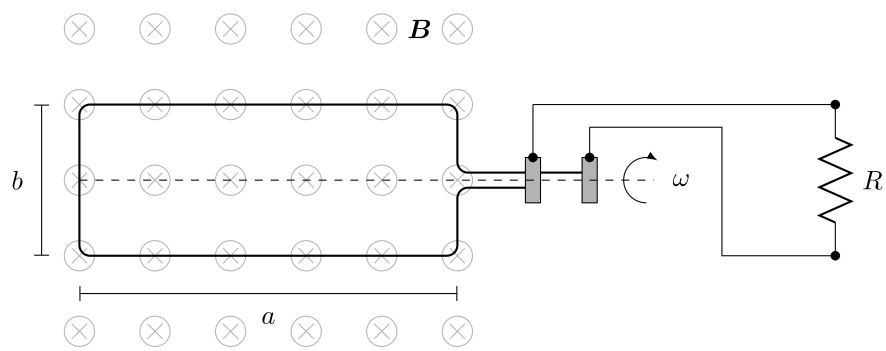
$$U = \frac{1}{2\mu_0} \int_{\text{space}} |\boldsymbol{B}|^2 \, dV \qquad \text{(Energy in the Magnetic Field)}$$

21.2 Problems

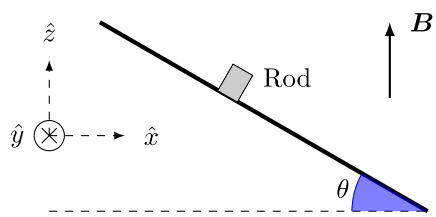
Problem 21.1: Alternating current generator. A rectangular loop of N turns with length a and width b is rotated at an angular frequency ω in a uniform field of induction $\boldsymbol{B}$ (See Figure.) Show that there is an induced EMF

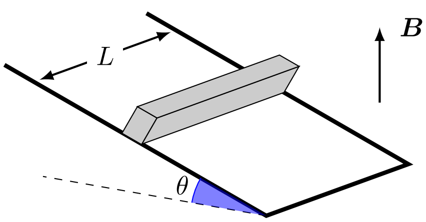
$$\mathcal{E} = \omega N b a B \sin \omega t = \mathcal{E}_0 \sin \omega t.$$

Source: Halliday and Resnick, Problem 35.9.



Problem 21.2: A rod with length ℓ, mass m, and resistance R slides without friction down parallel conducting rails of negligible resistance, as shown below. The plane of the rails makes an angle θ with the horizontal, and a uniform vertical magnetic field **B** exists throughout the region.





1. Find the steady-state terminal velocity of the sliding rod.
2. Show that the rate at which the internal energy of the rod is increasing is equal to the rate at which the rod is losing gravitational potential energy.
3. Discuss the situation if **B** were directed down instead of up.

Source: Halliday, Resnick, Krane 34.9

Problem 21.3: What is the energy dissipated as a function of time in a circular loop of N turns of wire having a radius of r and a resistance of R if the plane of the loop is perpendicular to a magnetic field given by

$$\boldsymbol{B}(t) = \boldsymbol{B}_0 e^{-t/\tau}.$$

Source: Giancoli, 29.72

21.3 Solutions

Solution 21.1: We apply Faraday's law. Let θ be the angle between the normal to the surface filling in the loop and the $\boldsymbol{B}$-field. Then $\theta(t) = \omega t + \theta_0$ for some arbitrary constant $\theta_0 \in \mathbb{R}$. Therefore the magnetic flux through all N loops is

$$\Phi_B(t) = N \int \boldsymbol{B} \cdot \hat{n}\, dA = N \int B \cos\theta(t)\, dA = NBA\cos\theta(t) Bab\cos(\omega t + \theta_0).$$

So by Faraday's law:

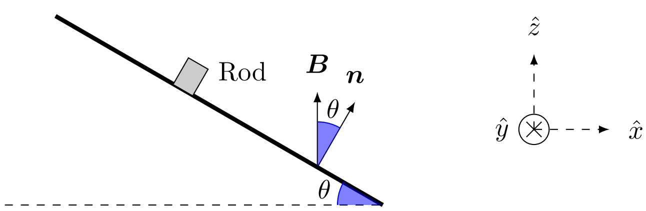
$$\mathcal{E}(t) = -\frac{d}{dt}\Phi_B(t) = -NBab\frac{d}{dt}\cos(\omega t + \theta_0) = NBab\omega\sin(\omega t + \theta_0) = \mathcal{E}_0 \sin\omega t.$$

So this will generate a sinusoidal EMF in the loop and hence an sinusoidal ("alternating") current. Depending on how we chose $\theta(t)$ we could have gotten $\pm\cos$ or $\pm\sin$ in the final answer. These are all equivalent and correspond to different choices of θ_0.

Solution 21.2:

1. There are many steps here. First we will use Faraday's Law to find the EMF generated in the wire. Second, we will use Ohm's law to determine the current in the wire. Third, we will use the Lorentz force law to determine the magnetic force on the rod. Next, we will draw a free-body diagram and decompose forces to find the equation of motion for the rod. Finally, we will apply the condition for terminal velocity (no acceleration) to find the terminal velocity of the rod.

   Let's consider a side view of the situation.



   Suppose that the distance between the bottom of the rails and the rod is $x(t)$. Then the magnetic flux is

$$\Phi_B(t) = \int \boldsymbol{B} \cdot \hat{n}\, dA = BA(t)\cos\theta = BLx(t)\cos\theta.$$

   By Faraday's law,

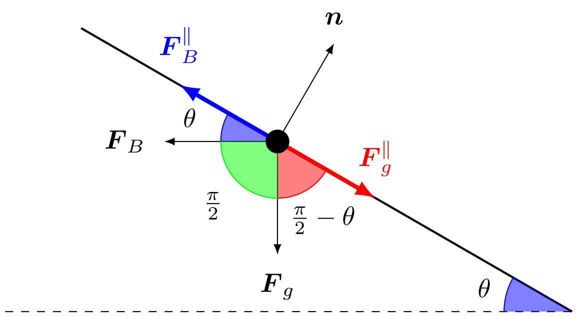
$$\mathcal{E} = -\frac{d\Phi_B(t)}{dt} = -BL\cos\theta\frac{dx}{dt} = -BL\cos\theta v.$$

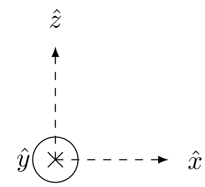
   The flux through the loop is decreasing so, by Lenz's law, the current created by the EMF wants to oppose thechange in flux and thus will be directed so that the induced magnetic field points along $\hat{n}$. This means the current must more towards us through the rod, i.e. $\boldsymbol{I} = \frac{\mathcal{E}}{R}\hat{y}$ (note that $\mathcal{E}$ is itself a negative quantity, so this points along the negative $\hat{y}$ direction).

   Knowing the current, we can calculate the force on the rod:

$$\boldsymbol{F}_B = \boldsymbol{I}\ell \times \boldsymbol{B} = \ell I\hat{y} \times B\hat{z} = -\ell IB\hat{x} = \ell\frac{-B\ell\cos\theta v}{R}B\hat{x} = -\frac{B^2\ell^2\cos\theta}{R}v\hat{x}.$$

   We now want to use a free-body diagram to find out where the rod will go.





There are three relevant forces: the magnetic force $\boldsymbol{F}_B$, gravity $\boldsymbol{F}_g = -mg\hat{z}$ and the normal force $\boldsymbol{N}$. The normal force will exactly cancel out the components of the magnetic and gravitational forces that try to pull the rod into the rails. To find the motion of the rod, we only need to know the components of the forces parallel to the rails. Since the normal force is perpendicular to the rails, we don't have to worry about it at all. Therefore the forces we care about are just those parallel to the rails:

$$F_{\|} = m\frac{dv}{dt} = F_B^{\|} + F_g^{\|}.$$

To find the parallel components, we simply project the force vectors along the rails. From the geometry of the diagram,

$$F_{\|} = m\frac{dv}{dt} = F_B^{\|} + F_g^{\|} = F_B \cos\theta + F_g \cos\left(\frac{\pi}{2} - \theta\right) = -\frac{B^2\ell^2\cos\theta}{R} v\cos\theta + mg\sin\theta.$$

The rod reaches terminal velocity when $\frac{dv}{dt} = 0$, so at

$$0 = -\frac{B^2L^2\cos\theta}{R} v\cos\theta + mg\sin\theta.$$

Solving for v yields

$$v = \frac{mgR\sin\theta}{B^2L^2\cos^2\theta}.$$

2. The internal energy of the rod is the energy of the current running through it. The rate at which that is increasing is the *power* of the that current:

$$P_{\text{elec}} = \frac{\mathcal{E}^2}{R} = \frac{1}{R}\left(BL\cos\theta v\right)^2 = \frac{B^2L^2\cos^2}{R} v^2.$$

Meanwhile, the gravitational energy of the rod is $U = mgh = mgx\sin\theta$, so its rate of change is

$$P_{\text{mechanical}} = \frac{dU}{dt} = mgv\sin\theta.$$

In the first part we found that at the steady-state,

$$0 = -\frac{B^2L^2\cos\theta}{R} v\cos\theta + mg\sin\theta.$$

Multiplying this by v shows $P_{\text{elec}} = P_{\text{mechanical}}$. During the non-steady state, we must also account for the change in kinetic energy of the rod.

3. If $\boldsymbol{B}$ is flipped, the flux is down and decreasing. To oppose this, the current will try to set up an induced magnetic field pointing down. This is done by a current travelling the *other* direction from part a), i.e. in the $\hat{y}$ direction along the rod.

   Suppose we started the rod going up the rails. Can you figure out a direction and time-dependence of the magnetic field that will make the rod accelerate without bound? This is the operating principle of the *rail gun*.

Solution 21.3: The energy begin dissipated by the loop is the power produced by the circuit. To find this, we first need to find the EMF. The magnetic flux through the circuit is

$$\Phi_B(t) = N\int \boldsymbol{B}\cdot\hat{n}\,dA = NB(t)\cos 0A = NAB_0e^{-t/\tau}.$$

Therefore

$$\mathcal{E} = -\frac{d\Phi_B(t)}{dt} = -NAB_0\frac{-1}{\tau}e^{-t/\tau}.$$

The power is then

$$P = \frac{\mathcal{E}^2}{R} = \frac{N^2A^2B_0^2e^{-2t/\tau}}{\tau^2R}.$$

22 Inductors

Inductors are components of electrical circuits that tend to *prevent changes in current.* When you try to change the current, the inductor will try to undo that change. In many ways, they are the magnetic analogues of capacitors. Inductors are characterized by their *inductance* L. The inductance is a strictly positive quantity which is determined entirely by the geometry — just like capacitance.

22.1 Helpful Equations

$$\begin{aligned}
\boldsymbol{F} &= q\boldsymbol{v}\times\boldsymbol{B}. && \text{(Coulomb's Law)}\\
d\boldsymbol{F} &= Id\hat{\ell}\times\boldsymbol{B} && \text{(Infinitesimal Coulomb's Law for a Wire)}\\
\oint_{\text{loop}}\boldsymbol{B}\cdot d\hat{\ell} &= \mu_0 I_{\text{enc}} && \text{(Ampère's Law)}\\
\boldsymbol{B}(\boldsymbol{r}) &= \frac{\mu_0}{4\pi}\int_{\text{current}}\frac{\boldsymbol{J}\times\hat{\boldsymbol{\imath}}}{\imath^2}\,dV = \frac{\mu_0}{4\pi}\int_{\text{wire}}\frac{\boldsymbol{I}\times\hat{\boldsymbol{\imath}}}{\imath^2}\,d\ell && \text{(Biot-Savart Law)}\\
\mathcal{E} &= \oint_{\text{loop}}\boldsymbol{E}\cdot d\boldsymbol{\ell} = -\frac{d\Phi_B}{dt} && \text{(Faraday's Law)}\\
U &= \frac{1}{2}LI^2 && \text{(Energy in an Inductor)}\\
U &= \frac{1}{2\mu_0}\int_{\text{space}}|\boldsymbol{B}|^2\,dV && \text{(Energy in the Magnetic Field)}
\end{aligned}$$

22.2 Problems

Problem 22.1: This problem will introduce inductors.

The simplest possible example of an inductor is a loop of wire. Specifically, suppose there is a circular loop of wire with a time-varying current $I(t)$.

1. Sketch the magnetic field in the plane of the loop.
2. Let's think about this situation. We have a current running through the loop, which gives us a field. The field goes through the loop, giving a non-zero magnetic flux. Using the definition of magnetic flux and then the Biot-Savart Law to find the $\boldsymbol{B}$ field, write the magnetic flux through the loop in the form

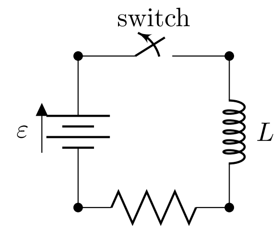
$$\Phi_B = LI(t)$$

   where L a quantity is in terms of several nasty integrals (don't actually do the integrals). The point is that the magnetic flux is *proportional* to the current.
3. Use Faraday's Law to show that the inductance is related to the induced emf $\mathcal{E}$ in the wire by

$$\mathcal{E} = -L\frac{dI}{dt} \text{ or } L = -\frac{\mathcal{E}}{\frac{dI}{dt}}. \tag{39}$$

   If L is greater than zero, then why does this equation need the minus sign on the right-hand side? (This is the equivalent of $C = Q/V$.)

   What's going on here physically? Suppose that the solenoid was in the following circuit.



When the switch is open, there is a steady current in going through the solenoid. This creates a uniform magnetic field in the center. Suppose the emf in the wire is suddenly reduced — the switch is opened. The current I will start to decrease at once; this *decrease* in current acts to *change* the magnetic field. Lenz's law ("natures hates changes in flux") tells us that the self-induction of the wire will oppose this change in current. To oppose the decreasing current, the induced emf points in the same direction as the current's original flow. Conversely, if the current is *increased* (perhaps by adding a second battery), the self-induced emf would point opposite to the direction of the current. In each case, the self-induction of the solenoid slows down the change in current. This is the setup of a LR circuit.

Problem 22.2: So far this isn't isn't very satisfying because, although we have an equation for L, it isn't very useful for calculating L.

Let's look at another situation where we can actually find L. Consider a solenoid of length ℓ and radius R where the wire makes n turns per unit length. (This is essentially a stack of $n \cdot \ell$ loops of wires, so it makes sense that it would also be an inductor.)

1. Using Ampére's Law, calculate the $\boldsymbol{B}$ field inside the solenoid.
2. Find the magnetic flux through *one* loop of the solenoid by integrating the last result. What is the *total* magnetic flux through the entire solenoid?
3. From the relation $\Phi_B = LI$, show that the inductance is
$$L = \mu_0 n^2 \ell (\pi R^2).$$
4. Calculate the magnetic energy stored in the solenoid by integrating the magnetic field. Use this to find the inductance.

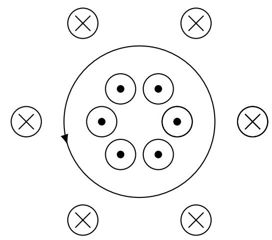
Problem 22.3: Suppose there is a toroid with a rectangular cross-section whose inner radius is a, outer radius is b, and height is h. Suppose that N turns of wire wrapped around the torus with current I going through them.

1. Use Ampère's Law to show that the magnetic field is
$$\boldsymbol{B} = \frac{\mu_0 I N}{2\pi r}\hat{\theta}.$$
2. Find the magnetic flux through one loop of the torus.
3. Show that the inductance of the toroid is
$$L = \frac{\mu_0 N^2 h}{2\pi} \ln \frac{b}{a}$$
using $\Phi_B = LI$
4. Calculate the energy stored in the inductor and show this satisfies $U = \frac{1}{2}LI^2$.

22.3 Solutions

Solution 22.1:

1. The magnetic field points up through the loop and down outside the loop.



I've drawn a circular loop here, but in principle this could be any shape.

2. The Biot-Savart law says the magnetic field inside the loop is

$$\boldsymbol{B} = \frac{\mu_0}{4\pi}\int_{\text{wire}} \frac{\boldsymbol{I}(t) \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\, d\ell = \frac{\mu_0}{4\pi} I(t) \int_{\text{wire}} \frac{\hat{I} \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\, d\ell$$

where $\hat{I}$ is the tangent vector to the wire and $I(t)$ is the strength of the current in the wire as a function of time. Therefore the flux throug the middle of the loop is

$$\Phi_B = \int_{\text{middle of the loop}} \boldsymbol{B}\cdot\hat{n}\, dA = \int \left(\frac{\mu_0}{4\pi} I(t) \int_{\text{wire}} \frac{\hat{I} \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\, d\ell\right)\cdot\hat{n}\, dA = I(t) \underbrace{\int \left(\frac{\mu_0}{4\pi} \int_{\text{wire}} \frac{\hat{I} \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\, d\ell\right)\cdot\hat{n}\, dA}_{L}\,.$$

So $\Phi_B = LI(t)$ for some constant L. Let's look at some general features of this expression. The integrals for L are quite nasty — this is not actually a great way to calculate the inductance. However, they do have one important quantity: they are completely geometric! By that, I mean that the value of L doesn't depend on anything to do with the actual current through the wire, only its shape. Furthermore, one can check that it is a strictly positive quantity by applying the right hand rule.

The inductance L is then *totally geometric* and *strictly positive*, just like capacitance!

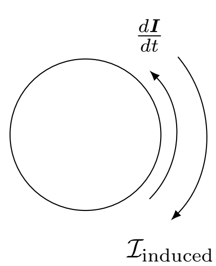
3. Faraday's law tells us that the EMF through the wire is

$$\mathcal{E} = -\frac{d\Phi_B}{dt}.$$

If $\Phi_B(t) = LI(t)$, then

$$\mathcal{E} = -\frac{d}{dt}LI(t) = -L\frac{dI}{dt}.$$

Let's use the right hand rule to examine what the sign here means physically.



The right hand rule tells us the that the EMF runs counter to the change in current. So when we *increase* the current in the loop, the EMF will create a current that runs in the opposite direction. This *usually* has the effect of partially cancelling out the change in current.

Solution 22.2:

1. We have done this problem already when we did Ampère's Law. The answer is

$$\boldsymbol{B} = \begin{cases} \mu_0 n I \hat{z} & \text{inside the solenoid} \\ \mathbf{0} & \text{outside the solenoid.} \end{cases}$$

   We assume that the solenoid is very long, so that the magnetic field of an infinite solenoid is a good approximation. This is true except at the ends of the solenoid, but if it's very long, those don't contribute much.

2. One loop of the solenoid is essentially a circle of radius R. Then

$$\Phi_B = \int \boldsymbol{B} \cdot \hat{n}\, dA = \int_0^{2\pi} \int_0^R \mu_0 n I \hat{z} \cdot \hat{z}\, r dr\, d\theta = 2\pi \mu_0 n I \frac{R^2}{2}$$

3. If $\Phi_B = \mu_0 n \pi R^2 I$ for a single loop, and there are $n\ell$ loops total, then

$$\Phi_{\text{total}} = \mu_0 n \pi R^2 I n \ell = \left(\mu_0 n^2 \ell \pi R^2\right) I = LI,$$

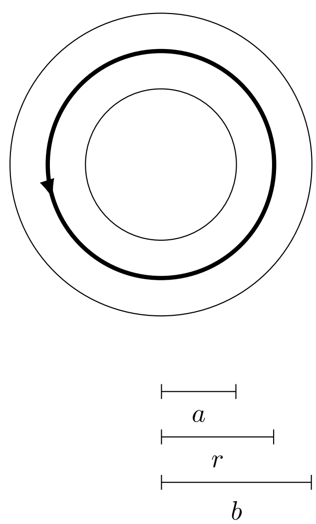
   so $L = \mu_0 n^2 \ell \pi R^2$. Note this is positive and totally geometric.

4. The energy density of a magnetic field is $\mathcal{U} = \frac{1}{2\mu_0} |\boldsymbol{B}|^2$. We want to integrate this up over the inside of the solenoid (since that's the only place the magnetic field is non-zero. Then

$$U = \frac{1}{2\mu_0} \int |\boldsymbol{B}|^2\, dV = \frac{1}{2\mu_0} \int_0^\ell \int_0^{2\pi} \int_0^R (\mu_0 n I)^2\, r dr\, d\theta\, dz = \frac{1}{2\mu_0} \left(\mu_0^2 n^2 I^2\right) 2\pi \frac{R^2}{2} \ell = \frac{1}{2} \left[\mu_0 n^2 \ell \pi R^2\right] I^2 = \frac{1}{2} L I^2.$$

Solution 22.3:

1. From the symmetry of the problem and the right hand rule, the magnetic field has the form $\boldsymbol{B} = B(r)\hat{\theta}$, i.e. the magnetic field will travel around clockwise inside the toroid. Let's take an Amperian loop going around the toroid at radius r. Consider the top view.



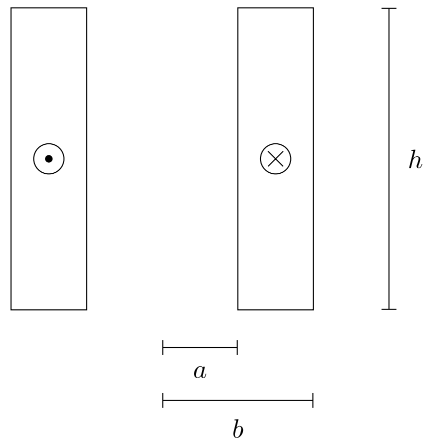
   Ampere's law says

$$\int_{\text{loop}} \boldsymbol{B} \cdot d\boldsymbol{\ell} = \int B(r)\, d\ell = B(r) 2\pi r = \mu_0 I_{\text{enc}}.$$

   There are N turns of wire on the toroid, and those all come up through the middle circle of radius a, so $I_{\text{enc}} = NI$ where I is the current in the wire. Therefore

$$\boldsymbol{B} = \frac{\mu_0 N I}{2\pi r}.$$

2. The magnetic field points around, so to find the flux through one view we need to integrate over the cross section of the toroid. Consider the side view.



Integrating over the rectangle gives a flux

$$\Phi_B = \int_{\text{rect}} \boldsymbol{B} \cdot \hat{n}\, dA = \int_a^b \int_0^h \frac{\mu_0 N I}{2\pi r}\, dr\, dz = \frac{\mu_0 N I}{2\pi} h \ln \frac{b}{a}.$$

3. There are N loops total, so

$$\Phi_{\text{tot}} = \frac{\mu_0 N I}{2\pi} h \ln \frac{b}{a} N = \left[\frac{\mu_0 N^2 h}{2\pi} \ln \frac{b}{a}\right] I.$$

Therefore $L = \frac{\mu_0 N^2 h}{2\pi}$.

4. Calculating the energy stored in the magnetic field is best done in cylindrical coordinates.

$$\begin{aligned}
U &= \frac{1}{2\mu_0} \int_0^h \int_0^{2\pi} \int_a^b B^2\, r dr\, d\theta\, dz \\
&= \frac{1}{2\mu_0} 2\pi h \int_0^h \left(\frac{\mu_0^2 N^2 I^2}{(2\pi)^2 r^2}\right) r\, dr \\
&= \frac{2\pi h \mu_0^2 N^2 I^2}{2\mu_0 (2\pi)^2} \int_0^h \frac{1}{r}\, dr \\
&= \frac{1}{2}\left[\frac{\mu_0 N^2 h}{2\pi} \ln \frac{b}{a}\right] I^2 \\
&= \frac{1}{2} L I^2.
\end{aligned}$$

23 LC and LRC Circuits

Now that we have a new circuit component, let's test it out in some circuits! Remember the rules for what the voltage across the different circuit elements are:

$$\mathcal{E}_R = IR, \quad \mathcal{E}_C = \frac{Q}{C}, \text{ and } \mathcal{E}_I = -L\frac{dI}{dt} \tag{40}$$

where $I = \frac{dQ}{dt}$ is the current and Q is the charge. The new twist now is that when we apply Kirchoff's rules, we'll now get a system of *differential* equations. For the most part, we'll content ourselves to stick to single-loop circuits. These are tough enough already!

23.1 Helpful Equations

$$\boldsymbol{F} = q\boldsymbol{v} \times \boldsymbol{B}. \qquad \text{(Coulomb's Law)}$$

$$d\boldsymbol{F} = I d\hat{\ell} \times \boldsymbol{B} \qquad \text{(Infinitesimal Coulomb's Law for a Wire)}$$

$$\oint_{\text{loop}} \boldsymbol{B} \cdot d\hat{\ell} = \mu_0 I_{\text{enc}} \qquad \text{(Ampère's Law)}$$

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \int_{\text{current}} \frac{\boldsymbol{J} \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\, dV = \frac{\mu_0}{4\pi} \int_{\text{wire}} \frac{\boldsymbol{I} \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\, d\ell \qquad \text{(Biot-Savart Law)}$$

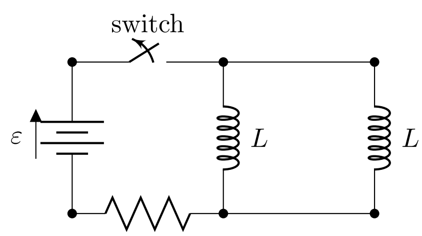
$$\mathcal{E} = \oint_{\text{loop}} \boldsymbol{E} \cdot d\boldsymbol{\ell} = -\frac{d\Phi_B}{dt} \qquad \text{(Faraday's Law)}$$

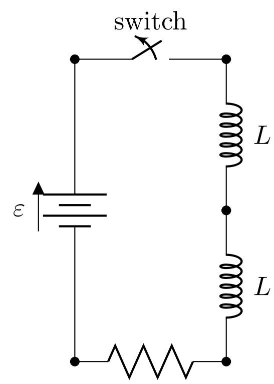
$$U = \frac{1}{2} L I^2 \qquad \text{(Energy in an Inductor)}$$

$$U = \frac{1}{2\mu_0} \int_{\text{space}} |\boldsymbol{B}|^2\, dV \qquad \text{(Energy in the Magnetic Field)}$$

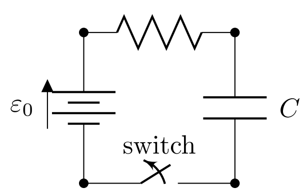
23.2 Problems

Problem 23.1: Suppose there are two inductors connected in series or parallel. Using Equation (39) and Kirchoff's rule, determine the equivalent inductance in each case.



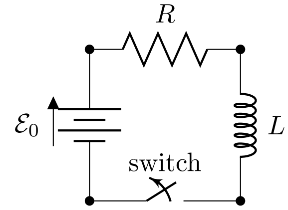


Problem 23.2: (RC Circuit) In the circuit shown, the capacitor is initially uncharged and the switch is initially open. There are a lot of questions here, but most of them are very quick — one line at most.



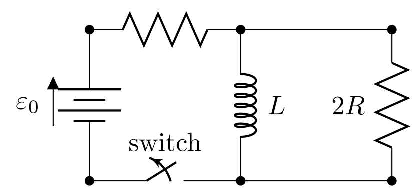
1. Immediately after the switch is closed, what is the voltage across the capacitor?
2. Immediately after the switch is closed, what current flows in the circuit?
3. What current flows a long time after the switched is closed?
4. A long time after the switch is closed, what is the voltage across the capacitor?
5. After a long time, how much charge has accumulated on the capacitor plates?
6. Using Kirchoff's law to find a differential equation for the current in the circuit.
7. If the switch is closed at $t = 0$, find $I(t)$ and sketch it. What is the time constant?
8. What is the energy stored in the capacitor at large times? How much of the total power dissipated by the circuit is stored there?

Problem 23.3: (LR Circuit) In the circuit shown, the capacitor is initially uncharged and the switch is initially open.

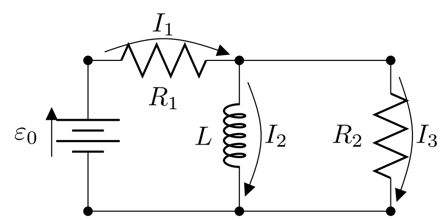


1. Immediately after the switch is closed, what current flows in the circuit?
2. Immediately after the switch is closed, what is the voltage across the resistor?
3. What current flows a long time after the switched is closed?
4. Using Kirchoff's law to find a differential equation for the current in the circuit.
5. If the switch is closed at $t = 0$, find $I(t)$ and sketch it. What is the time constant?

Problem 23.4: (More Complicated LR Circuit) In the circuit shown the switch is open and no currents are flowing.

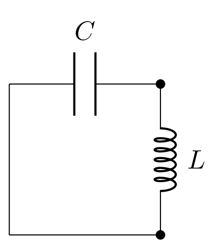


At $t = 0$ the switch is closed.



1. Immediately after the switch is closed, what are the currents I_1, I_2, and I_3?
2. After a long time, what are the currents?
3. Write down two loop rules and one junction rule and rearrange to get a differential equation for the current $I_2(t)$. Solve this explicitly.
4. After a very long time when the currents have reached the values in part b), the switch is reopened. Find the currents immediately afterwards.
5. Find the currents a very long time after that.
6. Find $I_2(t)$ explicitly when $t = 0$ is now the time when the switch was reopened.

Problem 23.5: (LC Circuit) At time $t = 0$, the capacitor is charged with charge Q_0.



1. Use Kirchoff's laws to write a differential equation for the current through the circuit.
2. Find the first time T when the capactor is fully discharged.

23.3 Solutions

Solution 23.1: Let I be the current going through the battery and let I_1 and I_2 be the currents through the inductors. In the parallel configuration, we know the voltage drop across both inductors must be the same, so

$$-L_1\frac{dI_1}{dt} = \mathcal{E} = -L_2\frac{dI_2}{dt}.$$

Integrating with respect to time gives $L_1 I_1 = L_2 I_2$, so $I = I_1 + I_2 = I_1\left(1 + \frac{L_1}{L_2}\right)$. Therefore the EMF across both inductors in parallel, which is the same as the EMF across either, is

$$\mathcal{E} = -L_{\parallel}\frac{dI}{dt} = -L_{\parallel}\frac{d}{dt}\left(I_1 + I_2\right) = -L_{\parallel}\frac{d}{dt}I_1\left[1 + \frac{L_1}{L_2}\right],$$

so by comparison,

$$L_{\parallel}\left(1 + \frac{L_1}{L_2}\right) = L_1 \implies L_{\parallel} = \frac{L_1}{1 + \frac{L_1}{L_2}} = \frac{L_1/L_1}{\left(1 + \frac{L_2}{L_1}\right)/L_1} = \frac{1}{\frac{1}{L_1} + \frac{1}{L_2}}.$$

There is probably a more elegant way to get this, but I can't think of it.

The series case is much easier. The current through both inductors is the same and the EMF through both is the sum of the EMFs of each, so

$$\mathcal{E} = -L_{\text{series}}\frac{dI}{dt} = \mathcal{E}_1 + \mathcal{E}_{\in} = -L_1\frac{dI}{dt} - L_2\frac{dI}{dt} = -\left(L_1 + L_2\right)\frac{dI}{dt}.$$

So inductances *add in series* and *add as reciprocals in parallel*, exactly like resistors.

Solution 23.2:

1. Immediately afterwards, there is no charge on the capacitor so by $Q = CV$, $V(t) = 0$ is zero across the capacitor.
2. Immediately afterwards, the capacitor acts like a wire as it charges up, so we effectively just have a resistor and a battery. Ohm's law then gives $\mathcal{E}_0 = IR$, so the current is simply $I(t=0) = \mathcal{E}_0/R$.
3. After a long time, the capacitor acts like a broken wire, so no current flows in the circuit.
4. After a long time, the capacitor is fully charged up. The battery has pushed as much charge onto it as possible. The equilibrium configuration when the battery can push no more charge onto it occurs when the voltage across the capacitor is equal to the battery voltage, so $\lim_{t\to\infty} V(t) = \mathcal{E}_0$ across the capacitor.
5. By $Q = CV$, $Q = C\mathcal{E}_0$.
6. Taking a loop going clockwise, Kirchoff's law says
$$\mathcal{E}_0 - IR + \frac{Q}{C} = 0.$$
7. Using $I = \frac{dQ}{dt}$,
$$\mathcal{E}_0 + R\frac{dQ}{dt} + \frac{Q}{C}$$
This equation is separable: we can put all the Q's on one side and the t's on the other:
$$R\frac{dQ}{dt} = -\left[\mathcal{E}_0 + \frac{Q}{C}\right] \implies \frac{RdQ}{Q + \mathcal{E}C} = -\frac{dt}{C} \implies \int_0^Q \frac{RdQ}{Q + \mathcal{E}C} = -\int_0^t \frac{dt}{C}$$
Integrating both sides gives
$$R\ln\left(Q + \mathcal{E}_0 C\right) = -\frac{t}{C} + D \implies \ln\left(Q + \mathcal{E}_0 C\right) = -\frac{t}{RC} + D$$

where D is an arbitrary constant to be set by initial conditions. Note that multiplying an arbitrary constant by anything is still an arbitrary constant, so $D/R = D$. Exponentiating both sides gives

$$Q + \mathcal{E}_{\prime}C = e^D e^{-t/RC} \implies Q(t) = -\mathcal{E}_0 C + e^D e^{-t/RC}.$$

We know from earlier that there is no charge on the capacitor at time zero,i.e. $Q(t=0) = 0$, so

$$0 = Q(0) = -\mathcal{E}_0 C + e^D e^0 = -\mathcal{E}_0 D + e^D \implies e^D = \mathcal{E}_0 C.$$

Therefore the charge on the capacitor is

$$Q(t) = \mathcal{E}_0 C \left[e^{-t/RC} - 1 \right].$$

To get the current in the circuit, we differentiate:

$$I(t) = \frac{dQ}{dt} = \mathcal{E}_0 C \frac{-1}{RC} e^{-t/RC} = -\frac{\mathcal{E}_0}{R} e^{-t/RC}.$$

You can check that this matches our predictions at early and late time from previous parts.

8. The capacitor stores energy $U = \frac{1}{2}CV^2 = \frac{1}{2}C\mathcal{E}_0^2$. The power dissipated by the resistor is

$$P(t) = \frac{dU}{dt} = I^2(t)R = R\left(-\frac{\mathcal{E}_0}{R}e^{-t/RC}\right) = \frac{\mathcal{E}_0^2}{R}e^{-2t/RC}.$$

Then the total work done by the resistor is

$$W = \int_0^\infty \frac{dU}{dt}\, dt = \int_0^\infty \frac{\mathcal{E}_0^2}{R} e^{-2t/RC}\, dt = \frac{\mathcal{E}_0^2}{R}\frac{RC}{2} = \frac{1}{2}C\mathcal{E}_0^2,$$

which is pretty neat. Why does this happen?

Solution 23.3:

1. Immediately afterwards, the inductor acts like a broken wire, so no current flows in the circuit.
2. Since there is no current in the circuit, there is no voltage drop across the resistor by Ohm's law.
3. After a long time, the inductor acts like a wire, so $\lim_{t\to\infty} I(t) = \frac{\mathcal{E}_0}{R}$.
4. Taking a clockwise loop,
$$\mathcal{E}_0 - IR - L\frac{dI}{dt} = 0.$$
5. If the switch it closed at $t = 0$, then $I(t = 0) = 0$. This is also a separable differential equation. Rearranging it we get
$$\frac{dI}{dt} = \frac{1}{L}\left[\mathcal{E}_0 - IR\right] = -\frac{R}{L}\left[I - \frac{\mathcal{E}_0}{R}\right] \implies \frac{dI}{I - \frac{\mathcal{E}_0}{R}} = -\frac{R}{L}dt.$$
Integrating both sides gives
$$\ln\left(I - \frac{\mathcal{E}_0}{R}\right) = -\frac{R}{L}t + D$$
where D is an arbitrary integration constant to be set by the initial condition. Exponentiating both sides gives
$$I(t) - \frac{\mathcal{E}_0}{R} = e^D e^{-\frac{t}{L/R}}.$$
With $0 = I(0)$, this becomes $-\mathcal{E}_0/R = e^D$, so
$$I(T) = \frac{\mathcal{E}_0}{R}\left[1 - e^{-t/(L/R)}\right].$$
The LR time constant is $\tau = L/R$.

Solution 23.4: The problem is probably too complicated to feature on an exam.

1. Immediately afterwards, the inductor acts like a broken circuit, so $I_2 = 0$ and $I_1 = I_3 = \mathcal{E}_0/(R_1 + R_2)$.
2. At long times, the inductor acts just like a wire so the electrons, wanting the path of least resistance, will go through the inductor and totally bypass the resistor R. Therefore in the limit, $I_1 = I_2 = \mathcal{E}_0/R_1$ and $I_3 = 0$.
3. Taking a clockwise path around each loop gives

$$\begin{aligned} 0 &= \mathcal{E}_0 - I_1 R_1 - L\frac{dI_2}{dt} \\ 0 &= L\frac{dI_2}{dt} - I_3 R_2 \\ I_1 &= I_2 + I_3. \end{aligned}$$

This is probably a more difficult set of differential equations than is reasonable to solve on an exam. However, it looks harder than it actually is because there's really only one equation in disguise. The first equation can be rearranged to

$$I_1 = \frac{\mathcal{E}_0}{R_1} + \frac{L}{R_1}\frac{dI_2}{dt}.$$

The third equation can be used to eliminate I_3 in the second equation, yielding

$$L\frac{dI_2}{dt} - (I_1 - I_2) R_2 = 0.$$

Combining the last two equations, we can eliminate I_1 entirely, giving

$$0 = L\frac{dI_2}{dt} - \left(\frac{\mathcal{E}_0}{R_1} + \frac{L}{R_1}\frac{dI_2}{dt} - I_2\right) R_2.$$

Rearranging,

$$\left(1 - \frac{R_2}{R_1}\right) L\frac{dI_2}{dt} = \frac{R_2}{R_1}\mathcal{E}_0 - I_2 R_2 = -R_2\left(I_2 - \frac{\mathcal{E}_0}{R_1}\right).$$

This is now a separable equation for I_2 of the type we've solved before, albeit with more complicated constants. Integrating it:

$$\left(1 - \frac{R_2}{R_1}\right)\int \frac{dI_2}{\left(I_2 - \frac{\mathcal{E}_0}{R_1}\right)} = -R_2 \int dt \implies \left(1 - \frac{R_2}{R_1}\right)\ln\left(I_2 - \frac{\mathcal{E}_0}{R_1}\right) = -R_2 t + D.$$

Rearranging gives

$$\ln\left(I_2 - \frac{\mathcal{E}_0}{R_1}\right) = -\frac{R_2 t}{1 - \frac{R_2}{R_1}} + D = \frac{-t}{\frac{1}{R_2} - \frac{1}{R_1}} + D.$$

Exponentiating both sides we get

$$I_2(t) = \frac{\mathcal{E}_0}{R_1} + e^D \exp\left(\frac{-t}{\frac{1}{R_2} - \frac{1}{R_1}}\right)$$

With the initial condition $I_2(t=0) = 0$ from earlier, we get $e^D = -\frac{\mathcal{E}_0}{R_1}$. Therefore

$$I_2(t) = \frac{\mathcal{E}_0}{R_1}\left[1 - \exp\left(\frac{-t}{\frac{1}{R_2} - \frac{1}{R_1}}\right)\right].$$

4. When the switch is opened, the circuit on the left-hand side is broken. Let this be the new $t = 0$. This means $I_1 = 0$ thereafter. Since inductors try to oppose changes in currents, when the current from the battery shuts off, the inductor will try to undo that change. The change is *against* I_2, as shown in the picture, so the inductor will create a *positive* current in the I_2 direction, which will slowly diminish to zero. In other words, the inductor will make the I_2 a continuous quantity. Immediately after the switch is reopened, $I_2 = \mathcal{E}_0/R_1$. Since that charge must flow somewhere, and it cannot flow through the left loop, it must flow through the right loop, so I_3 must be $-\mathcal{E}_0/R_1$.

5. A very long time after the switch is reopened, there is no battery providing power, so everything will have been dissipated by the resistor. At this point $I_1 = I_2 = I_3 = 0$.

6. By Kirchoff's law, $I_2 = -I_3$, so we may as well forget about I_3 and just use I_2. Then around the loop,

$$-L\frac{dI_2}{dt} - R_2 I_2 = 0 \implies \frac{dI_2}{I_2} = -\frac{R_2}{L}dt \implies \ln I_2 = -\frac{R_2}{L}t + D \implies I_2(t) = e^D \exp\left(-\frac{R_2}{L}t\right).$$

Using $I_2(0) = \frac{\mathcal{E}_0}{R_1}$ gives

$$I_2(t) = \frac{\mathcal{E}_0}{R_1}\exp\left(-\frac{R_2}{L}t\right).$$

Solution 23.5:

1. By Kirchoff's laws,

$$-L\frac{dI}{dt} - \frac{Q}{C} = 0.$$

Using $I = -\frac{dQ}{dt}$, this becomes

$$\frac{d^2Q}{dt^2} - \frac{1}{LC}Q = 0.$$

We immediately recognize this as the differential equation for a simple harmonic oscillator:

$$m\frac{d^2x}{dt^2} - kx = 0,$$

with $\frac{k}{m} = \frac{1}{LC}$. Thinking back to our mechanics knowledge, we can immediately write down the solution

$$Q(t) = A\cos\omega t + B\sin\omega t$$

where $\omega = \sqrt{\frac{1}{LC}}$. Because the capacitor has $Q(t=0) = Q_0$, and $Q'(t=0) = 0$, since the capactitor is not hooked up to anything as is about to start discharging, we find

$$Q_0 = Q(t=0) = A\cos\omega 0 + B\sin\omega 0 = A$$

and

$$0 = Q'(t=0) = -A\omega\sin\omega 0 + B\omega\cos\omega 0 = B\omega,$$

so $A = Q_0$ and $B = 0$. The charge on the capacitor as a function of time is therefore

$$Q(t) = Q_0\cos\sqrt{\frac{1}{LC}}t.$$

2. The first time the capactitor is fully dischanged is when

$$\cos\sqrt{\frac{1}{LC}}t = 0 \implies \sqrt{\frac{1}{LC}}t = \frac{\pi}{2} \implies t = \frac{\pi}{2}\sqrt{LC}.$$

24 AC Circuit Problems

Note: this material is only covered some years.

Extra-special bonus section this week! Today we're going to be looking at Alternating Current (AC) circuits! These are circuits that don't have a battery, but instead have an input voltage like $V(t) = V_0\cos\omega t$ that changes in time with some characteristic frequency ω. This is actually something you're already familiar with — all wall sockets provide alternating current. In the US, V_0 is 120 V and $\omega = 2\pi 60$ Hz.

There is a very slick technique for solving AC circuit problems: complex impedance. Using this technique, can convert the *differential* equations for circuits that we get from Kirchoff's law into *algebraic* equations — at a price. To do this, we need to work with complex-valued current, voltage and resistance. This is a pretty minor price to pay, in the end. Solving AC circuits ends up being easier than solving DC ones, so long as you remember a few facts about complex numbers.

There are two important caveats about complex impedance:

1. It only works for AC circuits where all voltage sources operate at the same frequency $\omega \neq 0$.

2. It only captures long-time behavior. When you first turn of the circuit, there are often signals that decay exponentially. These are called *transients*. The methods we're using will not capture these signals, so the answers you get will only be valid after they have decayed away.

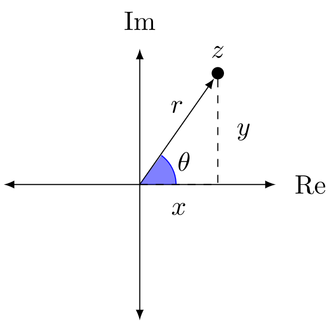
Both of these caveats can be fixed with more advanced techniques that are beyond the scope of this course. By using *Fourier Analysis*, one can work with circuits at multiple simultaneous frequencies, and by looking at *imaginary frequencies*, one can find the transients. Complex impedance is not a "trick," but it is a technique that only works in some situations — in particular, all the situations you will encounter in this class.

24.1 Complex Numbers

Before we dive into complex impedance, it's useful to review a few facts about complex numbers. I suspect everyone has seen these before, but it will save you a trip to Wikipedia. Complex numbers have two forms: Cartesian and polar. Any complex number z can represented as

$$z = x + iy \text{ or } z = re^{i\theta}. \tag{41}$$

A picture will let us figure out how to convert between the two.



By applying the pythagorean theorem and some basic trigonometry, we see that

$$r = \sqrt{x^2 + y^2}, \text{ and } \theta = \arctan\left(\frac{y}{x}\right). \tag{42}$$

(One should note that this is only strictly valid in the first quadrant. In other quadrants, θ has an extra minus sign or an extra π. A function usually called `atan2` is usually used to account for this in programming languages, or where the specific numerical value is important. We will use arctan and pass over this subtlety.) With this, we can convert between Cartesian and polar form for any imaginary number. Here are some additional useful facts:

1. Euler's Formula: $e^{i\theta} = \cos\theta + i\sin\theta$. This can also be seen from the geometry of the earlier picture. The horizontal axis is the "cosine axis" and the vertical axis is the "sine axis".

2. $\frac{1}{i} = -i$.

3. If $z = x + iy = r^{i\theta}$, then
$$\frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2} = \frac{1}{r}e^{-i\theta}. \tag{43}$$

4. The product of exponentials sum in the exponent: $e^{i\alpha}e^{i\beta} = e^{i(\alpha+\beta)}$.

I will use tilde's to denote complex versions of quantities.

24.2 Helpful Equations

$$
\begin{aligned}
I(t) &= \operatorname{Re}\tilde{I}(t) && \text{(Complex Current)}\\
\tilde{V} &= \tilde{I}Z && \text{(Complex Ohm's Law)}\\
Z_R &= R && \text{(Resistor Impedance)}\\
Z_C &= \frac{1}{i\omega C} && \text{(Capacitor Impedance)}\\
Z_L &= i\omega L && \text{(Inductor Impedance)}\\
Z_{\text{tot}} &= Z_1 + Z_2 && \text{(Impedance Series Law)}\\
\frac{1}{Z_{\text{tot}}} &= \frac{1}{Z_1} + \frac{1}{Z_2} && \text{(Impedance Parallel Law)}\\
P &= \langle I(t)V(t)\rangle = V_{\text{rms}} I_{\text{rms}} \cos\phi = \frac{V_0^2}{2\left|Z\right|}\cos\phi \text{ where } \tilde{I} = \left|\tilde{I}\right| e^{i\phi} && \text{(Power Dissipated by an AC Circuit)}
\end{aligned}
$$

Our general strategy for solving AC circuit problems with complex impedances is as follows.

1. Replace all currents by their complex counterparts and all resistances, capacitances, and inductances by their complex impedances. In particular, replace a voltage source

$$V(t) = V_0 \cos\omega t \longrightarrow \tilde{V}(t) = V_0 e^{i\omega t}.$$

   Similarly,

$$V(t) = V_0 \sin\omega t = V_0 \cos\left(\frac{\pi}{2} - \omega t\right) \longrightarrow \tilde{V}(t) = V_0 e^{i\left(\frac{\pi}{2} - \omega t\right)} = V_0 i e^{-i\omega t}.$$

2. Use Ohm's Law with the series and parallel formulas for impedance to solve for the quantity you care about. This works most of the time. Alternatively, one can use Kirchoff's rules, properly modified for complex currents:

$$\sum_{\text{loop}} \tilde{I}_k Z_k = 0 \text{ and } \sum_{\text{junction}} \tilde{I}_k = 0.$$

   This is usually only necessary when there are multiple voltage sources in the circuit.

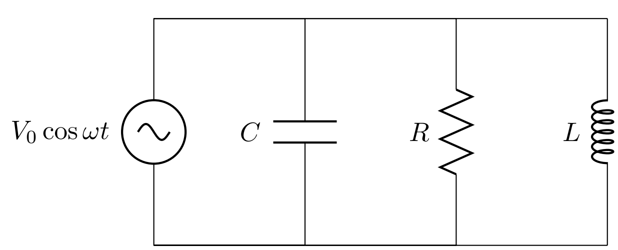
3. Convert all complex numbers in your answer to polar form.

4. Take the real part of your answer to go back to the "true" current, or use the Power Dissipated formula.

24.3 Problems

Let's try some problems.

Problem 24.1: (Parallel RLC Circuit) Consider the following circuit.

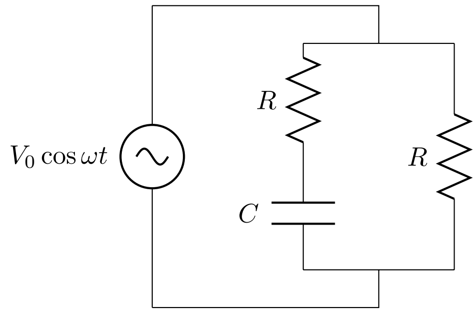


(a) Write Kirchoff's laws for the circuit, just to see how ugly it is. (This should be 5 equations).

(b) Find the complex impedance of the circuit and convert it to polar form $Z = Z_0 e^{i\phi}$.

(c) Find $I(t)$. Wasn't that much simpler than solving the differential equations?

Problem 24.2: (Two Resistors and a Capacitor) Consider the following circuit.



Here the frequency is chosen to be $\omega = \frac{1}{RC}$ (to simplify calculations).

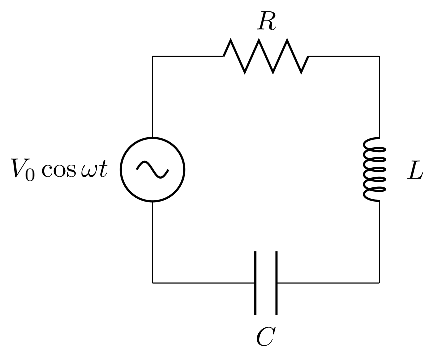
(a) What is the total complex impedance of the circuit? Give it in terms of R only.

(b) If the total current through the circuit is written as $I_0 \cos(\omega t + \phi)$, what are I_0 and ϕ?

(c) What is the average power dissipated in the circuit?

Source: Purcell, problem 8.38.

Problem 24.3: (All about Series RLC Circuits) Consider the following circuit.



(a) What is the total impedance of this circuit?

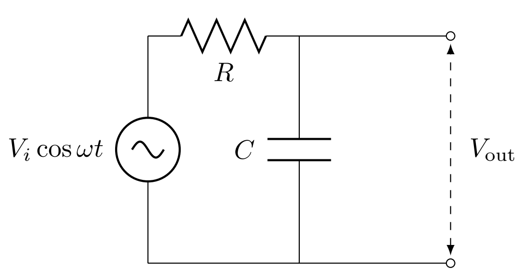
(b) Find $\tilde{I}$ coming out of the AC source in polar form. Take the real part to find the true current. What is it's amplitude? What is the phase difference between this and the input voltage?

(c) What is the earlier time t at which $Q(t)$, the charge on the capacitor, is zero?

(d) What is the resonant frequency of this circuit? If ω is at resonance, what is the amplitude and phase angle?

Source: 7B workbook.

Problem 24.4: (Low-Pass Filter) Consider the following circuit.This is a device known as a *low-pass filter*: its effect is to greatly attenuate high frequency signals. Let's see how it works!



Suppose we are driving a potential difference through the input on the left and we measure the voltage difference V_{out} on the right hand side.

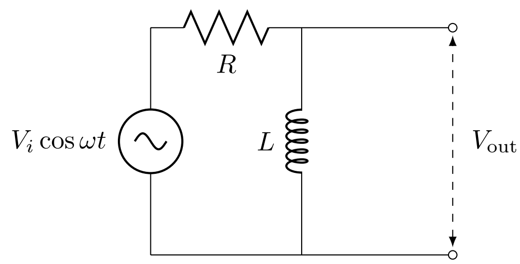
(a) Use Kirchoff's laws for complex impedance to show that

$$\frac{\tilde{V}_{\text{out}}(t)}{\tilde{V}_i(t)} = \frac{Z_C}{R + Z_C}.$$

(b) Find the magnitude of the right hand side as a function of ω and plot it.

Source: Purcell, problem 8.13.

Problem 24.5: (High-Pass Filter) Consider the following circuit.This is a device known as a *low-pass filter*: its effect is to greatly attenuate high frequency signals. Let's see how it works!

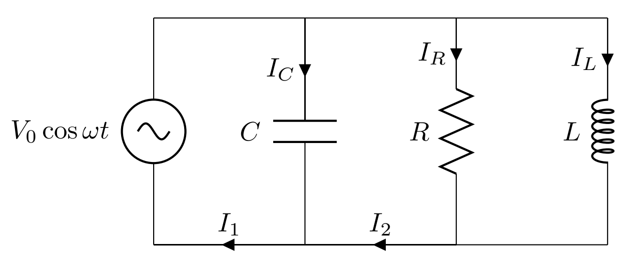


Suppose we are driving a potential difference through the input on the left and we measure the voltage difference V_{out} on the right hand side. Show this works as a high-pass filter that attenuates low frequency signals.

24.4 Solutions

Solution 24.1:

(a) Let the currents through the capacitor, resistor, and inductor be I_C, I_R and I_L respectively, as shown in the picture.



Then Kirchoff's laws say

$$\begin{aligned} 0 &= V_0 \cos \omega t - \frac{Q_C}{C} \\ 0 &= \frac{Q_C}{C} - I_R R \\ 0 &= I_R R - L\frac{dI_L}{dt} \\ I_1 &= I_2 + I_C \\ I_2 &= I_R + I_L. \end{aligned}$$

As you can see, these are pretty ugly and difficult to solve.

(b) The complex impedance is

$$\frac{1}{Z} = \frac{1}{Z_R} + \frac{1}{Z_C} + \frac{1}{Z_L} = \frac{1}{R} + i\omega C + \frac{1}{i\omega L} = \frac{1}{R} + i\left[\omega C - \frac{1}{\omega L}\right] = \sqrt{\frac{1}{R^2} + \left(\omega^2 C^2 - \frac{1}{\omega^2 L^2}\right)} \exp\left(i \arctan\left(\frac{\omega C}{R} - \frac{1}{\omega L R}\right)\right).$$

We will actually need $1/Z$ in the next step, so I'll leave it like this.

(c) The current $I(t)$ is the real part of the complex current:

$$\begin{aligned}
I(t) &= \operatorname{Re}\tilde{I}(t) \\
&= \operatorname{Re}\frac{\tilde{V}}{Z} \\
&= \operatorname{Re}\frac{V_0 e^{i\omega t}}{Z} \\
&= \operatorname{Re} V_0 \sqrt{\frac{1}{R^2} + \left(\omega^2 C^2 - \frac{1}{\omega^2 L^2}\right)} \exp\left(i \arctan\left(\frac{\omega C}{R} - \frac{1}{\omega L R}\right)\right) e^{i\omega t} \\
&= \operatorname{Re} V_0 \sqrt{\frac{1}{R^2} + \left(\omega^2 C^2 - \frac{1}{\omega^2 L^2}\right)} \exp\left(i\omega t + i \arctan\left(\frac{\omega C}{R} - \frac{1}{\omega L R}\right)\right) \\
&= V_0 \sqrt{\frac{1}{R^2} + \left(\omega^2 C^2 - \frac{1}{\omega^2 L^2}\right)} \cos\left(\omega t + \arctan\left(\frac{\omega C}{R} - \frac{1}{\omega L R}\right)\right).
\end{aligned}$$

In problems like these, it usually better to solve for the current in terms of the inductances and then only at the end actually find impedance as a complex number.

Solution 24.2:

(a) Using series and parallel laws:

$$\frac{1}{Z} = \frac{1}{Z_R} + \frac{1}{Z_R + Z_C} = \frac{1}{R} + \frac{1}{R - \frac{i}{\omega C}}.$$

Using the fact that $\omega = \frac{1}{RC}$,

$$\frac{1}{Z} = \frac{1}{R} + \frac{1}{R - i\frac{RC}{C}} = \frac{1}{R} + \frac{1}{R(1-i)}.$$

Since

$$1 + \frac{1}{1+i} = 1 + \frac{1}{1+i}\frac{1-i}{1-i} = \frac{2}{2} + \frac{1-i}{1+1} = \frac{3-i}{2}.$$

Therefore

$$\frac{1}{Z} = \frac{1}{R}\frac{3-i}{2} = \frac{1}{2R}\sqrt{3^2+1^2}e^{i\arctan\frac{-1}{3}} = \frac{\sqrt{10}}{2R}e^{i\arctan\frac{-1}{3}}$$

(b) The complex Ohm's law says $\tilde{V} = \tilde{I}Z$, so

$$I(t) = \operatorname{Re}\frac{\tilde{V}}{Z} = \operatorname{Re}\frac{V_0 e^{i\omega t}}{Z} = \operatorname{Re} V_0 \frac{\sqrt{10}}{2R} e^{i\arctan\frac{-1}{3}} e^{i\omega t} = \operatorname{Re}\frac{\sqrt{10}V_0}{2R} e^{i\left(\omega t + \arctan\frac{-1}{3}\right)} = \frac{\sqrt{10}V_0}{2R}\cos\left(\omega t + \arctan\frac{-1}{3}\right).$$

(c) The power dissipated is

$$P = \frac{I_0 V_0 \cos\phi}{2} = \frac{\sqrt{10}V_0}{2R} V_0 \underbrace{\cos\arctan\frac{-1}{3}}_{1/\sqrt{10}} = \frac{V_0^2}{2R}.$$

Solution 24.3:

(a) The total impedance is

$$Z = Z_R + Z_C + Z_L = R + \frac{1}{i\omega C} + i\omega L = R + i\left(\omega L - \frac{1}{\omega C}\right) = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} e^{i\arctan\left(\frac{\omega L}{R} - \frac{1}{\omega RC}\right)} = Z_0 e^{i\phi}$$

(b) The complex current is $\tilde{V} = \tilde{I}Z$, so

$$\tilde{I} = \frac{\tilde{V}}{Z} = \frac{V_0 e^{i\omega t}}{Z_0 e^{i\phi}} = \frac{V_0}{Z_0} e^{i(\omega t - \phi)}.$$

The "true" current is the real part:

$$I(t) = \operatorname{Re}\tilde{I} = \operatorname{Re}\frac{V_0}{Z_0} e^{i(\omega t-\phi)} = \frac{V_0}{Z_0}\cos(\omega t - \phi) = \frac{V_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \cos\left(\omega t - \arctan\left(\frac{\omega L}{R} - \frac{1}{\omega RC}\right)\right).$$

The amplitude is then

$$\frac{V_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

and the phase different is

$$\phi = -\arctan\left(\frac{\omega L}{R} - \frac{1}{\omega RC}\right).$$

(c) The charge on the capacitor is $Q = -\frac{dI}{dt}$, so

$$Q(t) = -\int_0^t I(t')\,dt' = -\int_0^t I_0 \cos(\omega t' - \phi)\ dt = -I_0 \frac{1}{\omega}\sin(\omega t - \phi).$$

This is zero when $\sin(\omega t - \phi) = 0$, i.e. $\omega t - \phi = \pi$ or $t = \frac{\pi + \phi}{\omega}$.

(d) The circuit is at resonance when

$$\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

is a minimum. This occurs when

$$\omega L - \frac{1}{\omega C} = 0 \implies \omega^2 LC = 1 \implies \omega_R = \frac{1}{\sqrt{LC}}.$$

There,

$$I(t) = \frac{V_0}{R}\cos\left(\omega t - \arctan\left(\frac{1}{R}\omega L - \frac{1}{R}\frac{1}{\omega C}\right)\right) = \frac{V_0}{R}\cos(\omega t - \arctan 0) = \frac{V_0}{R}\cos\omega t.$$

Solution 24.4:

(a) Let the current through the resistor be I_1 and through the capacitor be I_2. Then Kirchoff's laws for both loops says:

$$\begin{aligned} 0 &= \tilde{V}_i - \tilde{I}_1 R - \tilde{I}_2 Z_C \\ 0 &= \tilde{V}_0 + \tilde{I}_2 Z_C. \end{aligned}$$

Putting the second into the first equation gives

$$0 = \tilde{V}_i - \tilde{I}_1 R - \tilde{V}_0 \implies \tilde{V}_0 = \tilde{V}_i - \tilde{I}_1 R \implies \frac{\tilde{V}_i}{\tilde{V}_0} = 1 - \frac{\tilde{I}_1}{\tilde{V}_1} R.$$

We can recognize the ratio $\frac{\tilde{I}_1}{\tilde{V}_1} = \frac{1}{Z_1}$ where, by the series rule, $Z_1 = R + Z_C$. Therefore

$$\frac{\tilde{V}_i}{\tilde{V}_0} = 1 - \frac{R}{Z_1} = \frac{Z_1 - R}{Z_1} = \frac{R + Z_C - R}{R + Z_C} = \frac{Z_C}{R + Z_C}.$$

(b) Putting in the value for Z_C,

$$\frac{\tilde{V}_i}{\tilde{V}_0} = \frac{\frac{1}{i\omega C}}{R + \frac{1}{i\omega C}} = \frac{1}{i\omega CR + 1}.$$

Since $\operatorname{Re}\frac{1}{x+iy} = \operatorname{Re}\frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2}$,

$$\left|\frac{1}{i\omega CR + 1}\right| = \frac{1}{1 + (\omega CR)^2}.$$

Therefore

$$|V_0| = \sqrt{\tilde{V}_0^* \tilde{V}_0} = |V_i|\left(\frac{1}{\omega^2 C^2 R^2 + 1}\right).$$

Solution 24.5: This is exactly the same as before, except we make the substitution $Z_C \to Z_L$. The right-hand side is then

$$\frac{\tilde{V}_i}{\tilde{V}_0} = \frac{i\omega L}{R + i\omega L} = \frac{1}{\frac{R}{i\omega L} + 1} = \frac{1}{1 - i\frac{R}{\omega L}}.$$

The real part of this is

$$\frac{1}{1 + \frac{R^2}{\omega^2 L^2}}.$$

Therefore

$$|V_0| = |V_i| \left(\frac{1}{1 + \frac{R^2}{\omega^2 L^2}} \right).$$

One can check that this is zero as $\omega \to 0$ and 1 as $\omega \to \infty$.

25 Thermodynamics Review

Review questions that were given by GSIs in the whole-class review session in Fall 2016.

25.1 Review 1 (Saturday)

Problem 25.1: Consider a house with a square base a on each side and height h. The fraction of air that is :=N2 (molecular mass M_0) is $0 \le f \le 1$.

In this problem you may find the following useful:

- $1/(1 + x) \approx 1 - x$ for small x (be sure to justify using it).
- STP: $P = 1.01 \times 10^5$ Pa, $T = 273$ K.

(a) Estimate how many air molecules fill the interior space of the house when its temperature is T and the pressure is 1 atm. What assumptions have you made for the estimate?

(b) The walls of the house have a linear expansion coefficient α. If the temperature of the walls changes by $\Delta T > 0$, what is the fractional change in the interior volume?

(c) Assume that the house is *NOT* airtight, what is the fractional change in the number density of molecules (i.e. number per unit volume) corresponding to this temperature rise?

Source: Huang, Midterm 1, Spring 2011, Problem 1, `https://tbp.berkeley.edu/exams/3142/download/`.

Solution 25.1: See: `https://tbp.berkeley.edu/exams/3152/download/`.

Problem 25.2: A hot rock of mass $m = 10$ kg at a temperature $T = 1500$ K is placed into a container which is filled with 1 m^3 of water at a temperature of 300 K. The specific heat of water is approximately 4 kJ kg^{-1} K^{-1} and the specific heat of the rock is 1 kJ kg^{-1} K^{-1}. The density of water is 1 g m^{-3}.

(a) What is the final temperature of rock and water?

(b) What is the entropy change of the rock?

(c) What is the entropy change of the water?

(d) By how much did the height of the water change due to heating alone (ignoring the bigger effect of the addition of the rock), if the container had a depth of 2 m and a surface area of 0.5 m^2? The coefficient of volume expansion for water is 2×10^{-6} K^{-1}.

Source: Wurtele, Midterm 1, Spring 2013, Problem 2, `https://tbp.berkeley.edu/exams/4018/download/`.

Solution 25.2: See: `https://tbp.berkeley.edu/exams/3989/download/`.

Problem 25.3: An ideal gas undergoes adiabatic compression from $P_1 = 1.0\,\text{atm}$, $V_1 = 1\times10^6\,\text{L}$, $T_1 = 0\,°\text{C}$ to $P_2 = 1\times10^5\,\text{atm}$, $V_2 = 1\times10^3\,\text{L}$.

(a) Is the gas monatomic, diatomic, or polyatomic?

(b) What is the final temperature?

(c) How many moles of gas are in the sample?

(d) What is the total translational kinetic energy per mole before and after the compression?

(e) What is the ratio of the squares of the rms speeds before and after the compression?

Solution 25.3:

(a) Because this is adiabatic compression, we know $P_1V_1^\gamma = P_2V_2^\gamma$. Rearranging, $\frac{P_1}{P_2} = \left(\frac{V_2}{V_1}\right)^\gamma$. We want to solve for the exponent, so we take the log (base 10, for reasons that will be clear in a moment) of each side:

$$\log_{10}\frac{P_1}{P_2} = \gamma\log_{10}\frac{V_2}{V_1}.$$

But then

$$-5 = \log_{10}\frac{1}{10^5} = \log_{10}\frac{P_1}{P_2} = \gamma\log_{10}\frac{V_2}{V_1} = \gamma\log_{10}\frac{10^3}{10^6} = -3 \implies \gamma = \frac{5}{3} = \frac{3+2}{3}.$$

This implies the gas has three degrees of freedom and is monoatomic.

(b) Using $PV^\gamma = \text{constant}$ and the ideal gas law, one find $TV^{\gamma-1} = \text{constant}$, so

$$T_2 = \left(\frac{V_1}{V_2}\right)^{\gamma-1} T_1 = \left(10^3\right)^{\frac{5}{3}-1} T_1 = 100T_1 = 27\,300\,\text{K}.$$

(c) One can use $P_1V_1 = nRT_1$ and solve for n.

(d) From the equipartition theorem, the total kinetic energy is $K = \frac{3}{2}NkT = \frac{3}{2}nRT$, so

$$\frac{K_{\text{before}}}{n} = \frac{3}{2}rT_1 \text{ and } \frac{K_{\text{after}}}{n} = \frac{3}{2}rT_2.$$

(e) Since the kinetic energy is the average kinetic energy per molecule, times the number of molecules, and the rms speed is the speed which a molecule with average kinetic energy has, $K = N\frac{1}{2}mv_{\text{rms}}^2 = \frac{3}{2}nRT$. Therefore

$$\frac{v_{\text{rms,before}}}{v_{\text{rms,after}}} = \frac{T_1}{T_2} = \frac{1}{100}.$$

Problem 25.4:

(a) A rubber ball of mass m and radius r, and specific heat c is rolled off a cliff 10 m above the ground. It bounces back up to a height of 9 m. Assuming all the energy lost in the collision goes into the ball. During the collision, has heat been added to the ball? Has work been done on it? What is the temperature of the ball after the collision?

(b) If the ball always bounces to 90% of the starting height, what is the temperature of the ball after it stops bouncing? (*Hint*: there is way to do this without summing the series.)

(c) Suppose that we now take into account the expansion of the ball due to heating. If then linear expansion coefficient of the ball is α, then what is the final temperature?

Solution 25.4:

(a) From conversation of energy, 10% of the potential energy is converted into heat. So

$$0 = \Delta U = mc\Delta T - \frac{1}{10}mg(h-r) \implies \Delta T = \frac{mgh}{10mc} = \frac{gh}{10c}.$$

(b) After it stops bouncing all the gravitational potential has been converted to heat. Following the same logic as the last part, except without the factor of 1/10 gives $\Delta T = \frac{g(h-r)}{c}$.

(c) Conservation of energy gives

$$0 = \Delta U = mc\Delta T - mg(h - r - r\alpha\Delta T),$$

because the center of mass of the ball has risen slightly. One can solve for ΔT and see it will be slightly smaller than in the last part.

Problem 25.5: Consider a *parcel of air* moving to a different altititude z in the Earth's atmosphere. As the parcel changes altitude, it acquires the pressure $P(z)$ of the surrounding air. We have the equation

$$\frac{dP}{dz} = -\rho g \tag{44}$$

where $\rho(z)$ is the altitude-dependent mass density. During this motion, the pacel's volume will change and, because ais is a poor heat conductor, we assume this expansion or contraction will take place adiabatically.

(a) Starting with the equation $PV^\gamma =$ constant, show that for an ideal gas undergoing an adiabatic process, $P^{1-\gamma}T^\gamma =$ constant. Show the parcel's pressure and temperature are related by

$$(1-\gamma)\frac{dP}{dz} + \gamma\frac{P}{T}\frac{dT}{dz} = 0 \tag{45}$$

and thus

$$(1-\gamma)(-\rho g) + \gamma\frac{P}{T}\frac{dT}{dz} = 0. \tag{46}$$

(b) Use the ideal gas law with the result of the previous part to show that the change in the parcel's temperature with change in altitude is

$$\frac{dT}{dz} = \frac{1-\gamma}{\gamma}\frac{mg}{k_B}, \tag{47}$$

where m is the average mass of an air molecule, and k_B is the Boltzmann constant.

(c) Given that air is a diatomic gas ($d = 7$) with an average molecule mass of $29\,\mathrm{g\,mol^{-1}}$, show that $\frac{dT}{dz} = -9.8\,°\mathrm{C\,km^{-1}}$. This value is called the *adiabatic lapse rate* for dry air.

(d) In California, the prevailing westerly winds descend from one of the highest elevations (the 4000 m Sierra Nevada Mountains) to one of the lowest elevations (Death Valley, −100 m) in the continental US. If a dry wind has a temperature of −5 °C at the top of the Sierra Nevadas, what is the wind's temperature after it has descended to Death Valley?

Solution 25.5:

(a) From the ideal gas law,

$$\text{const} = PV^\gamma = P\left(\frac{NkT}{P}\right)^\gamma = P^{1-\gamma}T^\gamma (Nk)^\gamma \implies P^{1-\gamma}T^\gamma = \text{constant}.$$

Differentiating with respect to z,

$$0 = (1-\gamma)P^{\gamma-2}\frac{dP}{dz}T^\gamma + P^{1-\gamma}\gamma T^{\gamma-1}\frac{dT}{dz} = P^{\gamma-1}T^\gamma\left[(1-\gamma)\frac{dP}{dz} + \gamma\frac{dT}{dz}\right].$$

Dividing through yields

$$(1-\gamma)\frac{dP}{dz} + \gamma\frac{dT}{dz} = 0.$$

(b) Density is

$$\rho = \frac{M}{V} = \frac{mN}{V} = \frac{mP}{kT},$$

where M is the mass of the air per unit volume and m is the average mass of an air molecule. Therefore

$$(1-\gamma)(-\rho g) + \gamma \frac{dT}{dz} = 0 = (1-\gamma)\frac{mgP}{kT} + \gamma \frac{P}{T}\frac{dT}{dz} \implies \frac{dT}{dz} = \frac{1-\gamma}{\gamma}\frac{mg}{k_B}.$$

(c) Since this is diatomic at approximately room temperature, the number of degrees of freedom involve is 5, so $\gamma = \frac{7}{5}$. The rest is plugging in constants, which gives

$$\frac{dT}{dz} \approx -9.7727\,°\mathrm{C\,km^{-1}}.$$

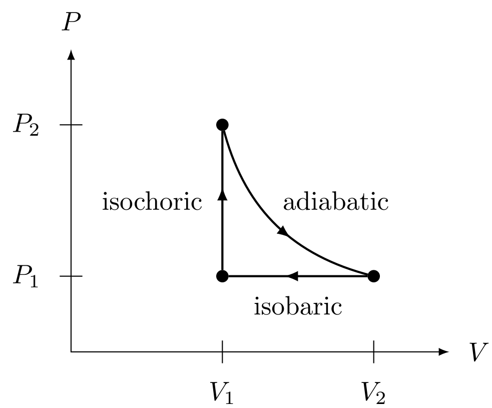
(d) Integrating the previous result gives

$$T(z) = \frac{\gamma - 1}{\gamma}\frac{mg}{k_B}z + C.$$

In this case, $T(z = 4000\,\mathrm{m}) = c = -5\,°\mathrm{C}$, so the temperature of the wind in death valley is

$$T(z = -100\,\mathrm{m}) = -9.7727\,°\mathrm{C\,km^{-1}} \times -4.1\,\mathrm{km} - 5\,°\mathrm{C} = 35\,°\mathrm{C} \approx 95°F.$$

Problem 25.6: Consider the following cycle for n moles of a monatomic ideal gas as the working substance. Express your answers for all parts in terms of n, P_1, P_2, and V_1 ONLY.



For parts (a) - (c), calculate the heat ΔQ that flows into, and the work W done by the gas. Remember to pay attention to the signs of ΔQ and ΔW.

(a) The adiabatic process.

(b) The isobaric process.

(c) The isochoric process.

(d) Find the net heat absorbed by the gas Q_{net}, and the net work done by the gas W_{net}, for the entire cycle.

(e) Find the efficiency of this cycle.

(f) Find the entropy change for the ideal gas for each of the processes. Please show your work for each process.

(g) Find the total entropy changes for the ideal gas for the entire cycle, ΔS, based on your calculations in the previous part. Is ΔS more or less than zero? Can you justify your answer?

Source: Huang, Midterm 1, Spring 2011, Problem 4, `https://tbp.berkeley.edu/exams/3142/download/`.

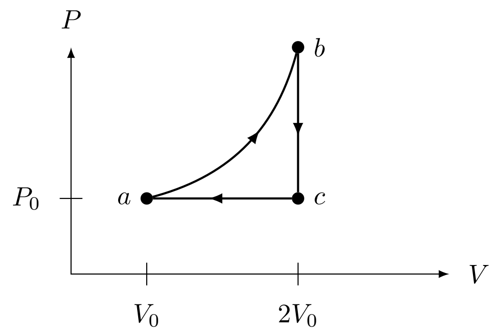
Solution 25.6: See: `https://tbp.berkeley.edu/exams/3152/download//`

25.2 Review 2 (Sunday)

Problem 25.7: [Heat Engines.] The figure shows a thermal cycle where, along the path from $a \to b$, the volume of a diatomic gas increases as

$$V(T) = V_0 \left(\frac{T}{T_a}\right)^s$$

where s is a constant and T_a is the temperature of the cas at point a in the cycle.



(a) Given the PV diagram shown, is $s > 1$ or $s < 1$? Why?

(b) What is the work done in the cycle? Express it in terms of s, V_0 and P_0.

(c) What is the efficiency of the cycle? Express it in terms of s.

Source: Speliotopoulos, Midterm 1, Fall 2014, Problem 4, `https://tbp.berkeley.edu/exams/4446/download/`.

Solution 25.7: See: `https://tbp.berkeley.edu/exams/4447/download/`.

Problem 25.8: [Thermal Processes.] For an adiabatic process, $đQ = 0$. The figure shows a gas of n moles of monoatomic molecules that undergoes a process for which

$$đQ = \frac{1}{2} đW.$$

(a) For each point along the PV diagram for the process, $PV^\beta = \text{constant}$. Using the first law of thermodynamics and the equation of state for the gas, determine β.

(b) If the temperatures at the beginning and end points of the process are T_a and T_b respectively, what is the change in entropy ΔS_{ab} for the process? Express it in terms of n,R, T_A and T_B.

Source: Speliotopoulos, Midterm 1, Fall 2014, Problem 5, `https://tbp.berkeley.edu/exams/4446/download/`.

Solution 25.8: See: `https://tbp.berkeley.edu/exams/4447/download/`.

Problem 25.9: [Second Law.] A hot meteorite of mass m, specific heat c, and initial temperature T_1 falls into the ocean whose temperature is T_2 ($T_1 = 3T_2$). Assume the ocean is so large that its temperature rise is negligible.

(a) Is the process reversible or irreversible? Explain.

(b) Determine the change in entropy of the meteorite.

(c) Determine the change in entropy of the surrounding environment.

(d) What is the total entropy change of the closed system? (Meteorite and ocean) Comment on the sign.

Source: Bordel, Midterm 1, Spring 2013, Problem 4, `https://tbp.berkeley.edu/exams/3973/download/`.

Solution 25.9: See: `https://tbp.berkeley.edu/exams/4180/download/`.

Problem 25.10: [A Fridge Problem.] The cooling compartment of a refrigerator operates at a low temperature of $T_L = -10.0\,°\mathrm{C}$, and exhausts heat into the air in the room at temperature $T_H = 21.0\,°\mathrm{C}$. The motor of the refrigerator requires $P = \frac{3}{4}$ horsepower of useful work to operate the refrigeration cycle. Give a *symbolic answer* in addition to a *numeric answer* for each portion of the question.

(a) What is the maximum possible coefficient of performance of this refrigerator?

(b) What is the rate, $\frac{dQ_L}{dt}$, that heat is taken out of the refrigerator, if it operates at a fraction $f = 43.0\%$ of its maximum coefficient of performance?

(c) What is the rate, $\frac{dQ_H}{dt}$, in which heat from the refrigerator is exhausted into the environment each second, if it operates at $f = 43.0\%$ of its maximum coefficient of performance?

(d) How long would it take to cool and freeze 4.20 kg of water at 18.0 °C to ice at 0 °C when placed in the refrigerator?

(e) What is the change in entropy of the ice, the air in the room, and the minimum change in the entropy of the universe during the freezing process?

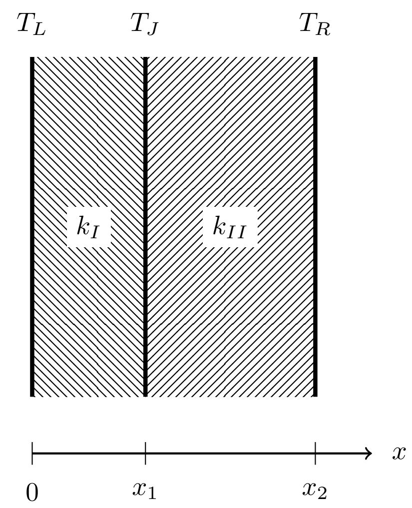
(f) What is the entropy change in the universe per cycle, if the refrigerator worked at the maximum possible coefficient of performance found in part (a)? Why? How much faster would it be from putting the water in until it freezes to ice? Give your answer as a fraction $t_{\mathrm{ideal}}/t_{\mathrm{actual}} = t_{100\%}/t_{43\%}$.

Source: Smoot, Midterm 1, Fall 2008, Problem 3, `https://tbp.berkeley.edu/exams/2757/download/`.

Solution 25.10: See: `https://tbp.berkeley.edu/exams/428/download/`.

Problem 25.11: [Conductive Heat Transfer]

Two slabs of different materials form a junction depicted in the figure, with a constant surface area A. The temperatures within the materials have reached equilibrium conditions. The boundaries at $x = 0$ and $x = x_2$ are held at constant temperatures T_L and T_R respectively. The two slabs are denoted by regions I and II, and the associated thermal conductivities are k_I and k_{II}, assumed constant within each material.



(a) What condition must be satisfied at the junction between materials I and II?

(b) Determine T_J in terms of known quantities.

(c) What is the rate of heat flow per surface area through region I? Give its unit.

(d) Now we consider this bilayer as a single layer of effective thermal conductivity k_{eff}. Calculate k_{eff} for a temperature difference $T_R - T_L = 30\,\mathrm{K}$, a total thickness $x_2 = 30\,\mathrm{cm}$, and a rate of heat flow per surface area of $10\,\mathrm{W\,m^{-2}}$.

Source: Bordel, Midterm 1, Fall 2012, Problem 4, `https://tbp.berkeley.edu/exams/3622/download/`.

Solution 25.11: See: `https://tbp.berkeley.edu/exams/3623/download/`.

Problem 25.12: Consider the following three-step process cycle: Heat is allowed to flow out of an ideal monatomic gas at constant volume so that its pressure drops from 2.2 atm to 1.5 atm. Then the gas expands at constant pressure, from a volume of 6.8 L to 10.0 L, where the temperature reaches its original value. The gas then moves along the isotherm back to its starting point.

(a) Draw a PV diagram showing the three process cycle.

(b) Calculate the total work done by the gas in the process cycle.

(c) Calculate the change in internal energy of the gas in the first two steps of the cycle.

(d) What is the heat flow into or out of the gas in each of the three steps?

(e) Is this a heat engine or refrigerator? What is its efficiency or coefficient of performance? Explain your answer.

(f) What is the entropy change for the whole world in one complete cycle? Why?

Source: Smoot, Midterm 1, Spring 2004, Problem 3, `https://tbp.berkeley.edu/exams/2758/download/`.

Solution 25.12: See: `https://tbp.berkeley.edu/exams/491/download/`.

26 Electricity Review

26.1 Electric Fields

The general strategy to solve electrostatics problems is as follows:

(a) Choose a coordinate system.

(b) Find dq in terms of geometric differentials (i.e. $dq = \lambda dx$).

(c) Choose an arbitrary small chunk of charge dq and find the differential electric field $d\boldsymbol{E}$ due to that chuck of charge. The tricky step here is using the geometry to determine $\boldsymbol{\mathscr{r}}$, the vector *from dq to the point* you're trying to compute the electric field at.

(d) Use superposition to write

$$\boldsymbol{E} = \int d\boldsymbol{E} = \int \frac{kdq}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}} \tag{48}$$

and integrate.

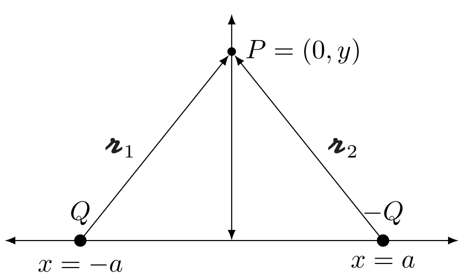
There's also a very useful trick that comes up often for dealing with unit vectors. The unit vector $\hat{\boldsymbol{\mathscr{r}}}$ is defined to be $\hat{\boldsymbol{\mathscr{r}}} = \boldsymbol{\mathscr{r}}/\mathscr{r}$, so the combination

$$\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} = \frac{1}{\mathscr{r}^2}\frac{\boldsymbol{\mathscr{r}}}{\mathscr{r}} = \frac{\boldsymbol{\mathscr{r}}}{\mathscr{r}^3}. \tag{49}$$

Electric Field due to Two Point Changes Suppose point charges Q and $-Q$ are placed on the x-axis at $x = -a$ and $x = a$ respectively. What is the electric field $\boldsymbol{E}$ along the y-axis? What is the electric field along the x axis?

Since we have no continuous charge distribution, we apply a slightly modified version of the four steps.

(a) Choose coordinates as follows.



(b) We have two discrete charges, Q and $-Q$.

(c) From the geometry of the situation,

$$\begin{aligned} \boldsymbol{\mathscr{r}}_1 &= a\hat{x} + y\hat{y} \\ \boldsymbol{\mathscr{r}}_2 &= -a\hat{x} + y\hat{y}. \end{aligned}$$

From the pythagorean theorem, $\mathscr{r}_1 = \mathscr{r}_2 = \sqrt{a^2 + y^2}$.

(d) Using the trick from Equation (49),

$$\boldsymbol{E} = \frac{kQ}{\mathscr{r}_1^2}\hat{\boldsymbol{\mathscr{r}}}_1 + \frac{k(-Q)}{\mathscr{r}_2^2}\hat{\boldsymbol{\mathscr{r}}}_2 = \frac{kQ}{\mathscr{r}_1^3}\boldsymbol{\mathscr{r}}_1 - \frac{kQ}{\mathscr{r}_3}\boldsymbol{\mathscr{r}}_2 = \frac{kQ}{(a^2+y^2)^{3/2}}\left((a\hat{x} + y\hat{y}) - (-a\hat{x} + y\hat{y})\right) = \frac{2kQ}{(a^2+y^2)^{3/2}}\hat{x}.$$

Along the x-axis, for $x > a$, the electric field is

$$\boldsymbol{E} = \frac{kQ}{\mathscr{r}_3^2}\hat{\boldsymbol{\mathscr{r}}}_3 + \frac{k(-Q)}{\mathscr{r}_4^2}\hat{\boldsymbol{\mathscr{r}}}_4 = \frac{kQ}{(x+a)^2}\hat{x} - \frac{k(-Q)}{(x-a)^2}\hat{x},$$

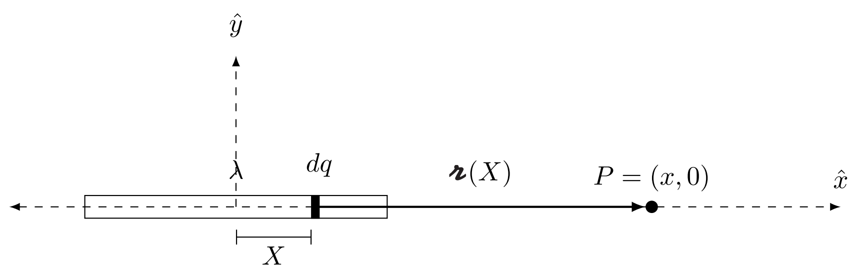
which doesn't simplify much.

Electric Field from a Line of Charge Suppose a rod of charge with linear charge density λ lies on the x axis between $x = -a$ and $x = a$. What is the electric field $\boldsymbol{E}$ at an arbitrary point on the x-axis to the right of the rod? At the point $(0, y)$?

We will assume that the rod has negligible thickness; approximate the rod by a line of charge.

First let's find the electric field on the x-axis.

(a) Choose coordinates as follows. Let the rod be parametrized by X running from $-a$ to a.



(b) The charge differential is

$$dq = \frac{\text{charge}}{\text{length}}\text{length} = \frac{Q}{2a}dX = \lambda dX.$$

(c) From the geometry, $\boldsymbol{\mathscr{r}}(X) = (x - X)\hat{x}$. Here we have used the fact that $x > a$. There would be a minus sign if $x < a$.

(d) Applying Equation (48),

$$\boldsymbol{E} = \int \frac{k\,dq}{\mathscr{r}(X)^2}\hat{\boldsymbol{\mathscr{r}}}(X) = \int_{-a}^{a} \frac{k\lambda dX}{(x-X)^2}\hat{x} = k\lambda\hat{x}\int_{-a}^{a}\frac{dX}{(x-X)^2}.$$

This integral can be done easily with the substitution $u = x - X$, and gives

$$\boldsymbol{E} = k\lambda\hat{x}\left.\frac{1}{x-X}\right|_{X=-a}^{X=a} = k\lambda\hat{x}\left(\frac{1}{x-a} - \frac{1}{x+a}\right).$$

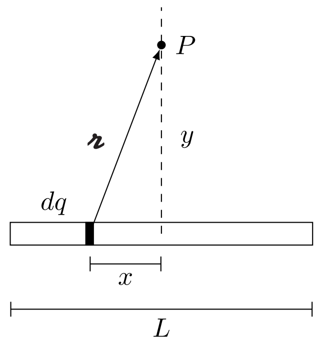
Now let's find the field on the y-axis.

(a) Let's choose coordinates where the origin is at the center of the rod, the x-axis is along the rod, and the y-axis is vertical.

(b) We will integrate along the x-axis from $x = -L/2$ to $x = L/2$. The charge dq is then

$$dq = \frac{\text{charge}}{\text{length}} = \frac{Q}{L}dx.$$

(c) Consider a chunk of charge dq at an arbitrary position x. This is shown in the picture below for the case $x < 0$.



From the geometry, we can immediate write $\boldsymbol{\mathscr{r}} = x\hat{x} + y\hat{y}$ (with $x < 0$ as drawn). Then the length of the vector is $\mathscr{r} = |\boldsymbol{\mathscr{r}}| = \sqrt{x^2 + y^2}$ by the Pythagorean theorem.

$$d\boldsymbol{E}(x) = \frac{kdq}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}} = \frac{kdq}{\mathscr{r}^2}\frac{\boldsymbol{\mathscr{r}}}{\mathscr{r}} = \frac{kdq}{\mathscr{r}^3}\boldsymbol{\mathscr{r}} = \frac{kdq}{(x^2+y^2)^{3/2}}(x\hat{x} + y\hat{y}) = \frac{k\frac{Q}{L}}{(x^2+y^2)^{3/2}}(x\hat{x} + y\hat{y})$$

(d) We can now integrate this to find the electric field:

$$\begin{aligned}
\boldsymbol{E} &= \int d\boldsymbol{E} \\
&= \int_{-L/2}^{L/2} \frac{k\frac{Q}{L}}{(x^2+y^2)^{3/2}}(x\hat{x} + y\hat{y}) \\
&= \frac{kQ}{L}\int_{-L/2}^{L/2} \frac{x}{(x^2+y^2)^{3/2}}\hat{x} + \frac{kQ}{L}\int_{-L/2}^{L/2} \frac{y}{(x^2+y^2)^{3/2}}\hat{y}.
\end{aligned}$$

the first integral is an easy u-substitution, and the second can be done with the trig substitution $x = y\tan\theta$, which gives

$$\begin{aligned}
&= \frac{kQ}{L}\left.\frac{-1}{\sqrt{x^2+y^2}}\right|_{-L/2}^{L/2}\hat{x} + \frac{kQ}{L}\left.\frac{x}{y\sqrt{x^2+y^2}}\right|_{-L/2}^{L/2}\hat{y} \\
&= \frac{kQ}{L}\left(\frac{-1}{\sqrt{\frac{L^2}{4}+y^2}} - \frac{-1}{\sqrt{\frac{L^2}{4}+y^2}}\right)\hat{x} + \frac{kQ}{L}\left(\frac{L/2}{y\sqrt{\frac{L^2}{4}+y^2}} - \frac{-L/2}{y\sqrt{\frac{L^2}{4}+y^2}}\right)\hat{y} \\
&= \frac{kQ}{y\sqrt{\frac{L^2}{4}+y^2}}\hat{y}.
\end{aligned}$$

Note that the $\hat{x}$ term cancelled out. We could have seen that this would happen way back at the beginning because of symmetry, made a note of that in step 3, and only done one of the integrals to speed things up.

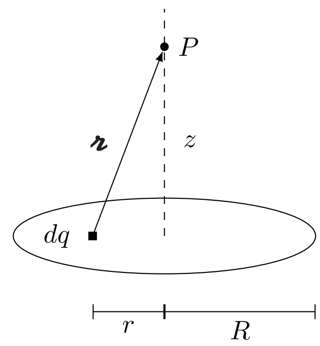
Electric Field from a Ring of Charge Suppose there is a ring of radius a lying in the xy-plane. Suppose the ring has charge Q distributed uniformly over its surface. What is the linear charge density λ? What is the electric field along the z-axis?

We use the same four steps as the problem above.

(a) Choose a coordinate system with the origin at the center of the ring. We will work in cylindrical coordinates with the z-axis perpendicular to the disk.

(b) Consider a chuck of the disk dq. Then $dq = \sigma dA = \sigma r dr d\theta$. (One can see that the area of a small section of a disk between r and $r + dr$ and between θ and $\theta + d\theta$ is change in radius $\times$ arclength $= (r - r + dr) \times r d\theta = r dr d\theta$, so $dA = r dr d\theta$ in cylindrical coordinates.)

(c) We draw the vector $\boldsymbol{\mathscr{r}}$ from dq to the point we can about. See the picture.



In cylindrical coordinates,

$$\boldsymbol{\mathscr{r}}(r,\theta) = -r\hat{r} + z\hat{z}.$$

Therefore

$$d\boldsymbol{E}(r,\theta) = \frac{kdq}{\mathscr{r}^3}\boldsymbol{\mathscr{r}} = \frac{k\sigma r dr d\theta}{(r^2+z^2)^{3/2}}\left[-r\hat{r} + z\hat{z}\right].$$

Before we integrate this, it pays to think for a second about symmetry. Because this is rotationally symmetric, the electric field will also be rotationall symmetric; the $\hat{r}$-component will cancel out when we integrate around the entire circle. Therefore we only have to worry about the z-component:

$$dE_z(r,\theta) = \frac{kq}{\mathscr{r}^3}\boldsymbol{\mathscr{r}} = \frac{kq}{(r^2+z^2)^{3/2}}z.$$

(d) Integrating over θ and r,

$$\begin{aligned}
\boldsymbol{E} &= \int dE_z(r,\theta)\hat{z} \\
&= \int_0^{2\pi}\int_0^R \frac{k\sigma r dr d\theta}{(r^2+z^2)^{3/2}} z\hat{z} \\
&= k\sigma z\left(\int_0^{2\pi} d\theta\right)\int_0^R \frac{rdr}{(r^2+z^2)^{3/2}}\hat{z} \\
&= \frac{1}{4\pi\varepsilon_0}\sigma z(2\pi)\left.\frac{-1}{(r^2+z^2)^{1/2}}\right|_0^R \hat{z} \\
&= \frac{\sigma}{2\varepsilon_0}z\left[\frac{-1}{\sqrt{R^2+z^2}} - \frac{-1}{\sqrt{z^2}}\right]\hat{z} \\
&= \frac{\sigma}{2\varepsilon_0}\left[1 - \frac{z}{\sqrt{R^2+z^2}}\right]\hat{z}.
\end{aligned}$$

Electric Field from a Plane of Charge Suppose there is an infinite plane of charge with uniform charge density σ. What is the electric field $\boldsymbol{E}$ at a height z above the plane? Below the plane?

This can be done in several equivalent ways: using Gauss's law with planar symmetry, calculating the potential and then integrating, or by taking the $R \to \infty$ limit in our last result. Let's do the last one. Note that

$$\lim_{R\to\infty}\frac{z}{\sqrt{R^2+z^2}} = 0,$$

so

$$\lim_{R\to\infty}\frac{\sigma}{2\varepsilon_0}\left[1 - \frac{z}{\sqrt{R^2+z^2}}\right]\hat{z} = \frac{\sigma}{2\varepsilon_0}\hat{z}.$$

It makes sense that the z-distance has dissappeared since, if you're looking at an infinite plane and you zoom in or out, everything looks exactly the same. This result will be very useful when we look at capacitors.

26.2 Electric Potentials

Electric potential problems are basically easier versions of electric field problems. Since the electric potential is a scalar, we don't have to worry about any of the vector geometry that made things complicated before. Moreover, because we only have a "$1/r$" instead of a "$1/r^2$" inside the integrals, most of the integrals are much easier. We can solve these problems with a modified and slightly easier 4-step strategy.

(a) Choose a coordinate system.

(b) Find dq in terms of geometric differentials (i.e. $dq = \lambda dx$).

(c) Choose an arbitrary small chunk of charge dq and find the differential electric field $d\boldsymbol{E}$ due to that chuck of charge. The tricky step here is using the geometry to determine $\mathscr{r}$, the **length** of the vector *from* dq *to the point* you're trying to compute the electric field at.

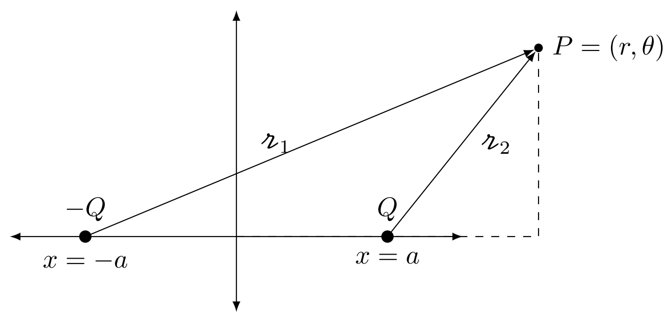
(d) Use superposition to write

$$V = \int \frac{kdq}{\mathscr{r}} \tag{50}$$

and integrate.

Potential due to Two Point Changes Suppose point charges Q and $-Q$ are placed on the x-axis at $x = a$ and $x = -a$ respectively. What is the potential V along the y-axis? Along the x-axis? What is the electric field from a dipole, i.e. this situation at a point (r, θ) where $r \gg d$.

(a) Choose polar coordinates with the origin between the two charges.



(b) We have two discrete charges Q and $-Q$.

(c) To find the lengths $\mathscr{r}_1$ and $\mathscr{r}_2$, we draw a line from (r, θ) to $(r\cos\theta, 0)$ on the x-axis (dashed line in the picture). Then the pythagorean theorem tells us

$$\mathscr{r}_1^2 = \left(r\cos\theta - \frac{d}{2}\right)^2 + (r\sin\theta)^2 = r^2\cos^2\theta + r^2\sin^2\theta - 2r\frac{d}{2}\cos\theta + \frac{d^2}{4} = r^2 + \frac{d^2}{4} - rd\cos\theta,$$

and

$$\mathscr{r}_2 = r^2 + \frac{d^2}{4} + rd\cos\theta,$$

a result we could also have gotten from the law of cosines.

(d) Then

$$V = \frac{kQ}{\mathscr{r}_1} - \frac{kQ}{\mathscr{r}_2} = kQ\left[\frac{1}{\sqrt{r^2 + \frac{d^2}{4} - rd\cos\theta}} - \frac{1}{\sqrt{r^2 + \frac{d^2}{4} + rd\cos\theta}}\right].$$

One can put in $\theta = 0$ or $\theta = \pi/4$ to get the cases of the x-axis or y-axis. In the general case, let us pull out a factor of r on the bottom:

$$V = \frac{kQ}{r}\left[\frac{1}{\sqrt{1 - \frac{d}{r}\cos\theta + \frac{1}{4}\frac{d^2}{r^2}}} - \frac{1}{\sqrt{1 + \frac{d}{r}\cos\theta - \frac{1}{4}\frac{d^2}{r^2}}}\right].$$

Note that $d \ll r$, so $\frac{d}{r} \ll 1$ is a very small quantity. We can then take a Taylor series in d/r. To first order, this gives

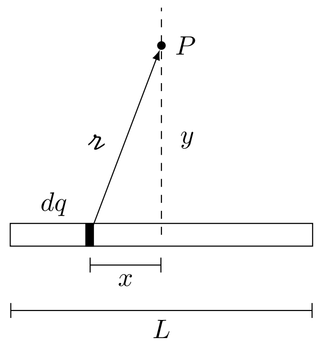
$$V = \frac{kQ}{r}\left[1 + \left(-\frac{1}{2}\right)(-1)\frac{d}{r}\cos\theta - 1 - \left(-\frac{1}{2}\right)(-1)\frac{d}{r}\cos\theta\right] = \frac{kQ}{r}\left[\frac{d}{r}\cos\theta\right] = \frac{kp\cos\theta}{r^2}$$

where $p = Qd$ is the dipole moment.

Potential from a Line of Charge Suppose a rod of charge with linear charge density λ lies on the x axis between $x = -L/2$ and $x = L/2$. What is the potential at an arbitrary point on the x-axis? At the point $(0, y)$?

First's lets find the potential at $(0, y)$.

(a) Let's choose coordinates where the origin is at the center of the rod, the x-axis is along the rod, and the y-axis is vertical.



(b) We will integrate along the x-axis from $x = -L/2$ to $x = L/2$. The chunks of charge are then $dq = \lambda dx$.

(c) By the pythagorean theorem, $\mathscr{r} = \sqrt{x^2 + y^2}$, so

$$dV = \frac{kdq}{\mathscr{r}} = \frac{k\lambda dx}{\sqrt{x^2 + y^2}}.$$

(d) Now we integrate:

$$V = \int_{-L/2}^{L/2} \frac{k\lambda dx}{\sqrt{x^2 + y^2}} = k\lambda \log\left(x + \sqrt{x^2 + y^2}\right)\Big|_{-L/2}^{L/2} = k\lambda \log \frac{\frac{L}{2} + \sqrt{\frac{L^2}{4} + y^2}}{-\frac{L}{2} + \sqrt{\frac{L^2}{2} + y^2}}.$$

On the axis at a point $X > L$, we find

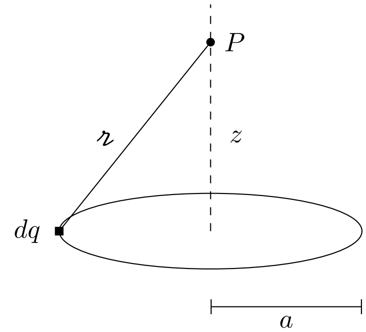
$$V = \int_{-L/2}^{L/2} \frac{k\lambda dx}{X - x} = k\lambda \log \frac{X - \frac{L}{2}}{X + \frac{L}{2}}.$$

Potential from a Ring of Charge Suppose there is a ring of radius a lying in the xy-plane. Suppose the ring has charge Q distributed uniformly over its surface. What is the linear charge density λ? What is the potential along the z-axis?

(a) Choose a coordinate system with the origin at the center of the ring. We'll use cylindrical coordinates with the z-axis perpendicular to the disk.

(b) We'll parameterize the ring by $0 \leq \theta \leq 2\pi$. A small angle $d\theta$ corresponds to an arclength $ad\theta$ of the ring, which has charge $dq = \lambda a d\theta$.

(c) Let the distance from dq to $(0, 0, z)$ be $\mathscr{r}$.



From the geometry, $\mathscr{r} = \sqrt{a^2 + z^2}$.

(d) Therefore the potential is

$$V = \int_0^{2\pi} \frac{k dq}{\mathscr{r}} = \int_0^{2\pi} \frac{k \lambda a d\theta}{\sqrt{a^2 + z^2}} = 2\pi \frac{k \lambda a}{\sqrt{a^2 + z^2}}.$$

Potential from a Plane of Charge Suppose there is an infinite plane of charge with uniform charge density σ. What is the potential at a height z above the plane? Below the plane?

This is best done using Gauss's law to find the electric field, $\boldsymbol{E} = \frac{\sigma}{2\varepsilon_0}\hat{z}$, for $z > 0$, and the integrating. I choose the zero-point of potential to be on the plane of charge, because taking it to be at infinity gives infinite results.

$$V(z) = -\int_0^z \boldsymbol{E}(z) \cdot \hat{z}\, dz = -\int_0^z \frac{\sigma}{2\varepsilon_0}\, dz = -\frac{\sigma z}{2\varepsilon_0}.$$

26.3 Dipoles

Lanzara 2013 Midterm 2, Problem 5 Two electric dipoles with dipole moments p_1 and p_2 are in line with one another. Assume the distance r between the dipoles is much greater than the length d of either dipole.

(a) Find the electric field due to an electric dipole at any position on the axis of the dipole.

(b) Find the potential energy U of one dipole in the presence of the other dipole (i.e. their interaction energy).

(c) If the dipoles are anti-aligned with one another, what is U?

(d) Now assume we turn on an electric field $\boldsymbol{E}_0$ away from p_1 towards p_2. Describe what happens to each dipole and find the maximum torque the field can exert on each dipole.

Solution: see `https://tbp.berkeley.edu/exams/4198/download/`.

26.4 Gauss's Law

Gauss's law gives a shortcut for computing the electric field in situations with high symmetry. The statement of Gauss's law is, given a 3-dimensional surface S, then the flux of the electric field passing through the surface is equal to the change enclosed by the surface:

$$\int_S \boldsymbol{E} \cdot \hat{n} dA = \frac{Q_{\text{enc}}}{\varepsilon_0}. \tag{51}$$

The left-hand side of this equation is a surface integral; you should read up on surface integrals in a multi-variable calculus book at some point during this course. However, to do these problems the only fact you need to know is that the surface integral of unity is the surface area:

$$\int_S 1\, dA = \text{surface area of } S.$$

Gauss's Law is can be proved using Green's theorem and is true in all cases. *However*, Gauss's Law is only *useful to us* in certain situations. Usually we are given the location and strengths of charges and we want to find the electric field. Normally Gauss's law is no help in doing this, because all it gives is the integral of the electric field. But, in situations with high symmetry, then we can pull the electric field outside the integral:

$$\int_S \boldsymbol{E}\cdot\hat{n}dA = E\int_S dA = E\times\text{surface area of } S \quad \text{(only true for with lots of symmetry)}.$$

In this case, we can compute the charge enclosed and solve for E. There are three cases were we can do this:

1. Spherical symmetry. Then (in spherical coordinates) $\boldsymbol{E} = E(r)\hat{r}$.
2. Cylindrical symmetry. Then (in spherical coordinates)$\boldsymbol{E} = E(r)\hat{r}$.
3. Planar symmetry. Then (in Cartesian coordinates) $\boldsymbol{E} = E(z)\hat{z}$ and $E(z) = -E(-z)$.

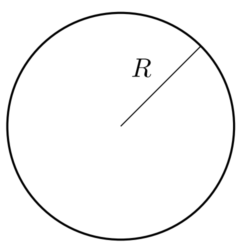
If we are in one of these special situations, we can use Gauss's law to determine $\boldsymbol{E}$ with the following procedure.

1. Identify the type of symmetry at play in this situation. Draw a "Gaussian surface", an invisible surface that respects the symmetry of the problem. For spherical symmetry, this is a sphere, etc.
2. Do the dot product inside the flux integral and show that, on each face of the surface, the dot product is either 0 or 1 and pull (a component of) the electric field outside the integral.
3. Compute the charge enclosed by the Gaussian surface.
4. Solve for (a component of) the electric field.

These steps can actually be very quick; one you have practice at Gauss's law, you can solve problems with it in 4-5 lines.

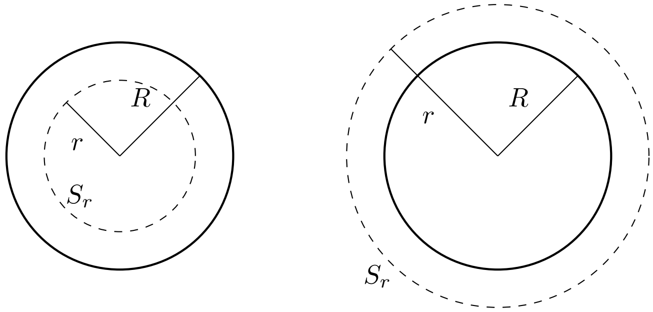
Let's dive into the problems now. As you practice these problems you'll get much quicker at them. To compensate for that, I've made each one uglier than the one before it.

Gauss's Law for a Spherical Shell Suppose there is a spherical shell (hollow ball) of radius R with charge Q spread uniformly on its surface. What is the surface charge density σ? What is the electric field $\boldsymbol{E}$ at all points in space? Keep in mind that $\boldsymbol{E}$ is a vector quantity.



Let's apply the four steps to find the electric field $\boldsymbol{E}$ at all points in space. There are two distinct cases: when $r < R$ and when $r > R$. We will therefore expect a piecewise function as answer.

1. This problem has spherical symmetry. Draw a Gaussian surface S_r, which is a sphere of radius r for some arbitrary r. There are two distinct cases: when $r < R$ and when $r > R$. We will therefore expect a piecewise function as answer.



$r < R$ $\qquad$ $r > R$

2. Due to the spherical symmetry, we know that the electric field has the form $\boldsymbol{E} = E(r)\hat{r}$. The outward unit normal vector to S_r is the radially outwards unit vector: $\hat{n} = \hat{r}$. Therefore

$$\int_{S_r} \boldsymbol{E}\cdot\boldsymbol{n}\, dA = \int_{S_r} E(r)\hat{r}\cdot\hat{r}\, dA = \int_{S_r} E(r) 1\, dA = E(r)\int_{S_r} 1\, dA = E(r)\times\text{surface area of } S_r = 4\pi r^2 E(r).$$

Here we were able to pull $E(r)$ outside the integral because the radius is a constant over S_r.

3. The charge enclosed by the Gaussian surface S_r is 0 when $r < R$ and Q when $r > R$. So

$$Q_{\text{enc}}(r) = \begin{cases} 0 & r < R \\ Q & r > R. \end{cases}$$

4. Gauss's law says

$$\int_{S_r} \boldsymbol{E}\cdot\hat{n} dA = \frac{Q_{\text{enc}}}{\varepsilon_0},$$

so using the result from parts 2 and 3 gives

$$4\pi r^2 E(r) = \frac{1}{\varepsilon_0}\begin{cases} 0 & r < R \\ Q & r > R. \end{cases}$$

so

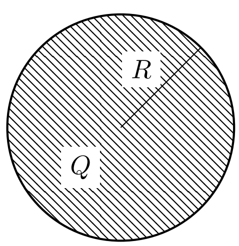
$$E(r) = \frac{1}{4\pi\varepsilon_0}\begin{cases} 0 & r < R \\ Q & r > R. \end{cases}.$$

Remembering that $\boldsymbol{E} = E(r)\hat{r}$, the electric field at all points in space is

$$\boldsymbol{E} = \frac{Q}{4\pi\varepsilon_0}\begin{cases} 0 & r < R \\ 1 & r > R. \end{cases}$$

Physically, this means that a uniformly charged sphere looks like a point charge when you're outside it. When you're inside it, however, the electric fields cancel each other out and you see no field at all.

Gauss's Law for a Sphere of Charge Suppose that there is a ball of radius R with charge Q uniformly distributed through the volume. What is the volume charge density ρ? What is the electric field $\boldsymbol{E}$ at all points in space?



This exactly the same except the charge enclosed is now different. The volume charge density is

$$\rho = \frac{\text{charge}}{\text{volume}} = \frac{Q}{\frac{4}{3}\pi R^3}.$$

For $r > R$, the charge enclosed is Q. For $r < R$, things are more tricky. Since the charge is *uniformly* distributed, the charge inside S_r will be

$$Q_{\text{enc}} = \frac{\text{charge}}{\text{volume}} \times \text{volume} = \rho\frac{4}{3}\pi r^3 = \frac{Q}{\frac{4}{3}\pi R^3}\frac{4}{3}\pi r^3 = Q\frac{r^3}{R^3}.$$

Therefore

$$4\pi r^2 E(r) = \int_{S_r} \boldsymbol{E}\cdot\hat{n} dA = \frac{Q_{\text{enc}}}{\varepsilon_0} = \frac{1}{\varepsilon_0}\begin{cases} Q\frac{r^3}{R^3} & r < R \\ Q & r > R \end{cases}$$

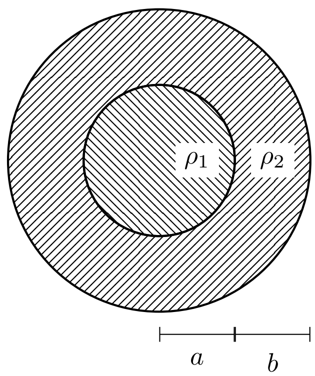
so, solving for $E(r)$ and adding back in the unit vectors,

$$\boldsymbol{E} = E(r)\hat{r} = \frac{1}{4\pi\varepsilon_0}\begin{cases}\frac{1}{r^2}Q\frac{r^3}{R^3} & r < R \\ \frac{1}{r^2}Q & r > R\end{cases}\hat{r} = \frac{Q}{4\pi\varepsilon_0}\begin{cases}\frac{r}{R^3} & r < R \\ \frac{1}{r^2} & r > R.\end{cases}\hat{r}.$$

So the electric field increases linearly until it gets to the surface, and then falls off like $1/r^2$. From the outside, a spherical shell (the last problem) and a ball of charge (this problem) are indistinguishable.

Gauss's Law for a Non-Uniform Sphere Suppose there is a ball of radius a with uniform volume charge density ρ_1. Suppose this is surrounded by a spherical shell of thickness b with uniform volume charge density ρ_2.

1. Write the volume charge density $\rho(r)$ as a piecewise function of r, the radial distance.
2. What is the electric field $\boldsymbol{E}$ at all points in space?



The volume charge density is

$$\rho(r) = \begin{cases}\rho_1 & 0 < r < a \\ \rho_2 & a < r < b \\ 0 & r > b.\end{cases}$$

We can apply exactly the same argument as before to find the electric field; the only difference is the charge enclosed. For $0 < r < a$, this is the same as the last problem. For $a < r < b$, the charge enclosed is the charge inside radius a plus the charge between a and b:

$$Q_{\text{enc}} = \rho_1\frac{4}{3}\pi a^3 + \rho_2\left(\frac{4}{3}\pi r^3 - \frac{4}{3}\pi a^3\right) = (\rho_1 - \rho_2)\frac{4}{3}\pi a^3 + \rho_2\frac{4}{3}\pi r^3.$$

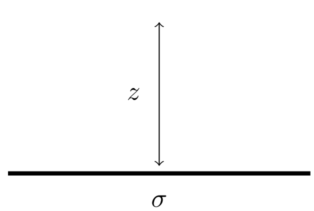
When $r > b$, then the amount of charge is just the total charge:

$$Q_{\text{enc}} = (\rho_1 - \rho_2)\frac{4}{3}\pi a^3 + \rho_2\frac{4}{3}\pi b^3$$

Therefore

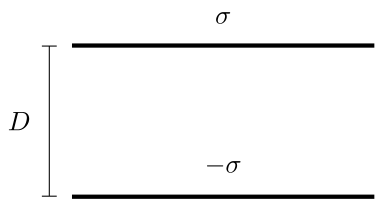
$$\boldsymbol{E} = E(r)\hat{r} = \frac{1}{4\pi r^2}\frac{1}{\varepsilon_0}\begin{cases}\rho_1\frac{4}{3}\pi r^3 & 0 < r < a \\ (\rho_1 - \rho_2)\frac{4}{3}\pi a^3 + \rho_2\frac{4}{3}\pi r^3 & a < r < b \\ (\rho_1 - \rho_2)\frac{4}{3}\pi a^3 + \rho_2\frac{4}{3}\pi b^3 & b < r\end{cases} = \frac{1}{3\varepsilon_0}\hat{r}\begin{cases}\rho_1 r & 0 < r < a \\ (\rho_1 - \rho_2)\frac{a^3}{r^2} + \rho_2 r & a < r < b \\ (\rho_1 - \rho_2)\frac{a^3}{r^2} + \rho_2\frac{b^3}{r^2} & r > b.\end{cases}$$

Gauss's Law for a Plane Suppose there is an infinite plane of change of negligible thickness with a uniform surface change density σ. What is the electric field at a distance z away?



This problem is solved in the textbook.

Gauss's Law for Parallel Plates Suppose there is an infinite plane of change of negligible thickness with a uniform surface change density σ and, a distance d away, there is another infinite plane with surface charge density $-\sigma$. Calculate $\boldsymbol{E}$ at all points in space



There are several ways to do this problem

- Use above result of the electric field above a disk of charge after taking the limit $R \to \infty$ and then use superposition to find the electric field.
- Use Gauss's law to find the electric field for an infinite plane of uniform charge and then use superposition. We cannot use Gauss' law for both planes at once because, when you rotate everything by π, the situation is not the same because $\sigma \neq -\sigma$. Therefore we need to do the planes one at a time.

Let's do the first option because it's the easiest and you've probably seen the second one in the textbook or in class. From above we have that the electric field for an infinite plane of uniform charge density σ is

$$\boldsymbol{E} = \frac{\sigma}{2\varepsilon_0}\,\mathrm{sgn}(z)\hat{z}.$$

If you haven't encountered it before, the "sign function" tells you what the sign of a quantity is:

$$\mathrm{sgn}(x) = \begin{cases} +1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0. \end{cases}$$

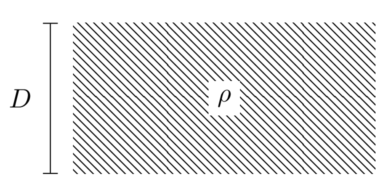
In this case, this tells us the electric field points up on top of the plane and points down beneath it. This is in fact a general fact whenever there is planar symmetry: the electric field has the form $\boldsymbol{E} = E(z)\hat{z}$ where $E(z) = -E(-z)$. $E(z)$ is an odd function because, if you rotate the entire situation by π, then it must be the same.

For both planes at once, we use superposition to find

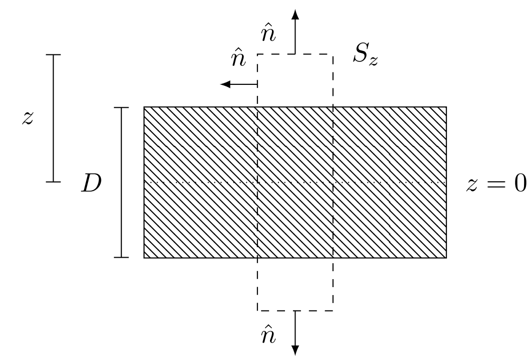
$$\boldsymbol{E} = \frac{\sigma}{2\varepsilon_0}\,\mathrm{sgn}\left(z + \frac{D}{2}\right)\hat{z} + \frac{-\sigma}{2\varepsilon_0}\,\mathrm{sgn}\left(z - \frac{D}{2}\right)\hat{z} = \frac{\sigma}{\varepsilon_0}\hat{z}\begin{cases} 1 & |z| < D/2 \\ 0 & |z| > D/2. \end{cases}$$

So the electric field is constant and vertical inside and zero outside. We will use this as a model for a capacitor in the near future.

Gauss's Law for a Slab Suppose there is an infinite slab of thickness D with uniform volume charge density ρ. Calculate the electric field at all points in space.



Imagine the Gaussian surface S_z, a cylinder with height $2z$ and area A on the top and bottom. We don't need a cylinder in particular; any surface with a straight edges whose top and bottom are flat and the same shape would work just as well. There are two cases: $z > D/2$ (shown in picture) and $z < D/2$.



First let's find the flux. The outward pointing unit normal to S_z points in different directions on the various faces of S_z. On the top it point in the $\hat{z}$ direction, on bottom it points in the $-\hat{z}$ direction and on the sides it points in the $\hat{r}$ direction, which is perpendicualr to $\hat{z}$. Therefore

$$\begin{aligned}\int_{S_z} \boldsymbol{E}\cdot\hat{n}\,dA &= \int_{\text{top}} \boldsymbol{E}\cdot\hat{n}\,dA + \int_{\text{bottom}} \boldsymbol{E}\cdot\hat{n}\,dA \int_{\text{sides}} \boldsymbol{E}\cdot\hat{n}\,dA \\ &= \int_{\text{top}} E(z)\hat{z}\cdot\hat{z}\,dA + \int_{\text{bottom}} E(-z)\hat{z}\cdot(-\hat{z})\,dA \int_{\text{sides}} E(z)\hat{z}\cdot\hat{r}\,dA \\ &= E(z)\int_{\text{top}} dA - E(-z)\int_{\text{bottom}} +0\end{aligned}$$

but $E(z) = -E(-z)$ and the surface area of the top and bottom A, so

$$= 2E(z)A.$$

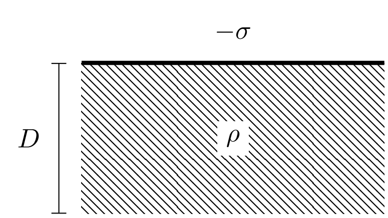
Meanwhile the charge enclosed by S_z is $\rho(2z)A$ when $z < D/2$ and ρDA for $z > D/2$. Therefore

$$2E(z)A = \int_{S_z} \boldsymbol{E}\cdot\hat{n}\,dA = \frac{1}{\varepsilon_0}\begin{cases}\rho DA & 0 \le z > D/2 \\ \rho 2zA & z < D/2\end{cases}$$

so the electric field for the slab is

$$\boldsymbol{E} = E(z)\hat{z} = \frac{1}{2A\varepsilon_0}\,\mathrm{sgn}(z)\hat{z}\begin{cases}\rho 2zA & |z| < D/2 \\ \rho DA & 0 \le |z| > D/2\end{cases} = \frac{\rho}{\varepsilon_0}\,\mathrm{sgn}(z)\hat{z}\begin{cases}2z & |z| < D/2 \\ D & |z| < D/2.\end{cases} \tag{52}$$

Gauss's Law for Parallel Plates and a Slab Suppose there is an infinite plane of charge of negligible thickness with a uniform surface charge density $-\sigma$ and, next to it, an infinite slab of thickness D with uniform volume charge density ρ. Calculate the electric field at all points in space.



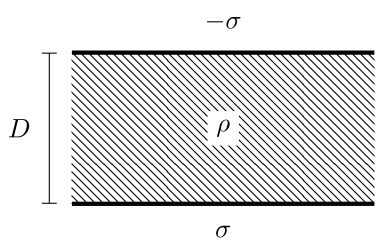
We cannot directly apply Gauss's law because the sitaution is not symmetric under a flip $z \mapsto -z$. However, we can find the electric field for just the slab using Gauss's law and then add in the plane with superposition.

The electric field for the whole situation is therefore

$$\boldsymbol{E} = \boldsymbol{E}_{\text{slab}} + \boldsymbol{E}_{\text{plane}} = \frac{\rho}{\varepsilon_0}\,\mathrm{sgn}(z)\hat{z}\begin{cases}2z & |z| < D/2 \\ D & |z| < D/2.\end{cases} + \frac{-\sigma}{2\varepsilon_0}\,\mathrm{sgn}\left(z + \frac{D}{2}\right)\hat{z}.$$

We can combine this into a piecewise function, but it's not particularly nice, so I haven't bothered.

Gauss's Law for Parallel Plates and a Slab Suppose we have the same setup as the last problem, but add a conducting slab of charge σ below the slab. Now what is $\boldsymbol{E}$ everywhere in space?



We can get this just using superposition of the previous results:

$$\begin{aligned}\boldsymbol{E} &= \boldsymbol{E}_{\text{slab}} + \boldsymbol{E}_{\text{top plane}} + \boldsymbol{E}_{\text{bottom plane}} \\ &= \frac{\rho}{\varepsilon_0}\operatorname{sgn}(z)\hat{z}\begin{cases} 2z & |z| < D/2 \\ D & |z| < D/2. \end{cases} + \frac{-\sigma}{2\varepsilon_0}\operatorname{sgn}\left(z+\frac{D}{2}\right)\hat{z} + \frac{\sigma}{2\varepsilon_0}\operatorname{sgn}\left(z-\frac{D}{2}\right)\hat{z}.\end{aligned}$$

Gauss's Law for a Cylinder Suppose there is an infinitely long cylinder with radius a and surface charge density Q. What is the electric field everywhere in space?

Solution: let r be the radial distance from the center of the cylinder. Then

$$\boldsymbol{E}(r) = \begin{cases} 0 & r < a \\ \dfrac{\sigma}{\varepsilon_0}\dfrac{a}{r}\hat{r} & r > a \end{cases}$$

Gauss's Law for a Uniformly Charged Cylinder Suppose there is an infinitely long cylinder of radius b. The charge density of the cylinder is ρ for $0 \leq r \leq a$. What is the electric field at all points in space?

Solution: let r be the radial distance from the center of the cylinder. Then

$$\boldsymbol{E}(r) = \begin{cases} \dfrac{\rho r}{2}\hat{r} & r < a \\ \dfrac{\rho a^2}{2r}\hat{r} & r > a \end{cases}$$

26.5 Capacitors

Parallel Plate Capacitor What is the capacitance of a parallel plate capacitor? Precisely, suppose there are two parallel plates with surface area A, constant separation d, and charges Q and $-Q$ respectively. What happens if we double the separation? Double the area? How does this relate to capacitors in series and parallel?

Solution:

$$C = \frac{\varepsilon_0 A}{d}.$$

Spherical Capacitors Suppose there are two spherical shells at radii a and b. Suppose they carry charges Q and $-Q$ respectively. What is the capacitance?

Solution:

$$C = \frac{ab}{a-b} = \frac{4\pi\varepsilon_0}{\frac{1}{b}-\frac{1}{a}}.$$

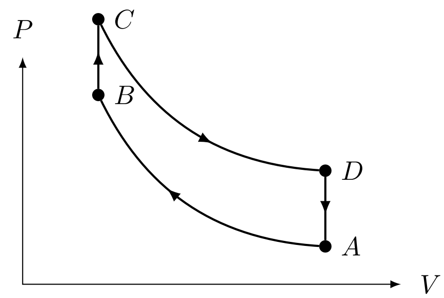
Cylindrical Capacitor Suppose there are two cylindrical shells at radii a and b with length $\ell \gg a, b$. Suppose they have charges Q and $-Q$. What is the capacitance?

Solution:

$$C = \frac{2\pi\varepsilon_0\ell}{\ln(b/a)}.$$

27 Final Review

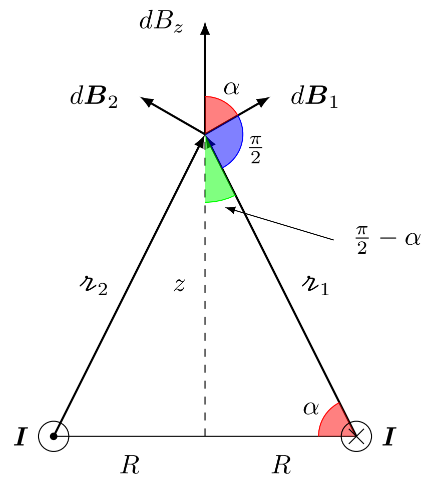
Problem 27.1: (Bordel, Midterm 1, Fall 2012.) The operation of an automobile internal combustion engine can be approximated by a reversible cycle known as an Otto cycle, which involved two adiabatic paths and two isovolumetric paths. Assume we are using n moles of an ideal *diatomic* gas as the working substance and assume the system is not hot enough to have appreciable vibrational kinetic energy.



1. Express the work done by the gas along each branch of the cycle in terms of pressures and volumes.
2. What is the amount of heat flowing into the gas associated with each of the four processes in this cycle in terms of pressures and volumes.?
3. What is the change in entropy of the gas along each branch?
4. Compute the efficiency of the cycle in terms of V_a and V_b only.

Solution: See `https://tbp.berkeley.edu/exams/3623/download/`, question 5.

Problem 27.2: (Magnetic Field above a Loop) Suppose there is a circular wire loop of radius R. If the loop carries a steady current I, what is the magnetic field a distance z above the center? Be careful with your geometry here! *Source: Griffiths EM, Example 5.6.*
Solution.
Consider a side view of the problem.



Considering the small part of the wire on the right-hand bottom, the cross-product will give a vector $d\boldsymbol{B}_1$. However, the contribution from the other side of the circle will give $d\boldsymbol{B}_2$. These will clearly cancel out when we integrate around the circle, so we only need to keep track of the vertical component. Therefore we want to compute:

$$dB_Z = \frac{\mu_0 I}{4\pi} \frac{\left(d\hat{\ell} \times \hat{r}\right)_z}{r^2}.$$

First,

$$\left| d\hat{\ell} \times \hat{r} \right| = \sin\frac{\pi}{2} d\ell = d\ell$$

so the z-component is the projection along the z-axis:

$$\left(d\hat{\ell} \times \hat{r}\right)_z = (d\ell)_z = \cos\alpha d\ell = \frac{R}{r} d\ell.$$

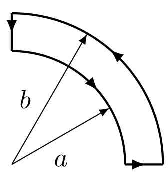
From the pythagorean theorem, $r = \sqrt{R^2 + z^2}$, so

$$\begin{aligned} dB_Z &= \frac{\mu_0 I}{4\pi} \frac{\left(d\hat{\ell} \times \hat{r}\right)_z}{r^2} \\ &= \frac{\mu_0 I}{4\pi} \frac{1}{r^2} \frac{R}{r} d\ell \\ &= \frac{\mu_0 I}{4\pi} \frac{R}{r^3} R d\theta \\ &= \frac{\mu_0 I}{4\pi} \frac{R^2}{\left(R^2 + z^2\right)^{3/2}} d\theta. \end{aligned}$$

Integrating this gives

$$B = \int_0^{2\pi} dB_z = \frac{\mu_0 I}{4\pi} \frac{R^2}{\left(R^2 + z^2\right)^{3/2}} 2\pi = \frac{\mu_0 I}{2} \frac{R^2}{\left(R^2 + z^2\right)^{3/2}}.$$

Problem 27.3: (Magnetic Field from a Quarter Circle) Suppose there is a wire in the shape shown below with a steady current I travelling clockwise. What is the magnetic field at point P?



Solution.

This can be done re-using old problems. The two straight sides of the wire are parallel or anti-parallel to the $\hat{r}$ vector, so the cross-products vanish there. They therefore do not contribute to the magnetic field. Using the right-hand rule, both semi-circles contribute a component in the vertical direction. To find the magnitudes of those components, it is easiest to find the magnetic field due to a quarter circle of arbitrary radius and current: $\boldsymbol{B}_{\text{Q.C.}}(R, I)$.

However, a quarter circle is just a quarter of a loop, so we can just use dB_z from above, with $z = 0$. Therefore

$$\boldsymbol{B}_{\text{Q.C.}}(R, I) = \int_0^{\pi/2} \frac{\mu_0 I}{4\pi} \frac{R^2}{\left(R^2 + 0^2\right)^{3/2}} d\theta = \frac{\mu_0 I}{8R}.$$

The total magnetic field is then:

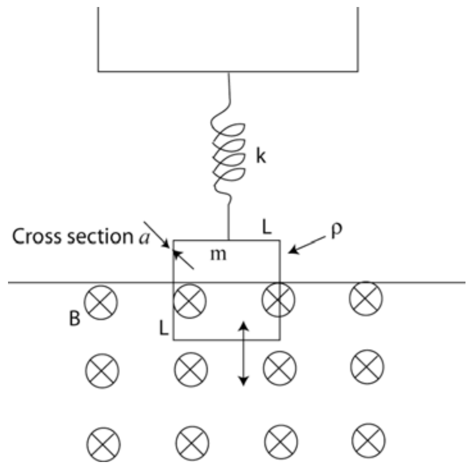
$$\boldsymbol{B} = \boldsymbol{B}_{\text{Q.C.}}(a, I) + \boldsymbol{B}_{\text{Q.C.}}(b, -I) = \frac{\mu_0 I}{8} \left(\frac{1}{a} - \frac{1}{b}\right).$$

Problem 27.4: (Bordel, Final 2012) At $t = 0$, a square coil of side $2d$ and resistance R is placed in a perpendicular time-dependent uniform magnetic field of magnitude $B = B_0(1 - kt)$ where $k > 0$ is a positive constant. No current initially passes through the coil.

1. Calculate the EMF $\mathcal{E}$ induced in the loop.
2. Calculate the induced current and explain its direction using Lenz's law.
3. Calculate the magnetic field B_i created by one side of the current carrying loop at its center.
4. Calculate the total magnetic field B_{tot} at the center of the loop.

Solution. See: `https://tbp.berkeley.edu/exams/4580/download/`, problem 6.

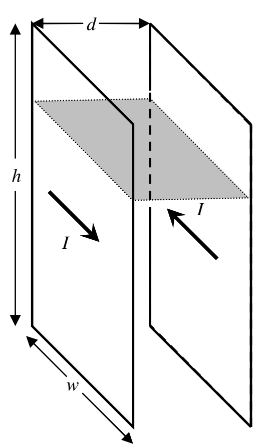
Problem 27.5: (Packard, Final Fall 2004) In physics 7A you learned that a damped simple harmonic oscillator (mass M, spring constant k, frequency $\omega = \sqrt{k/m}$) experiences a linear friction force $F_f = -b\dot{x}$ where $\dot{x}$ is the velocity of the mass. The transient response of the oscillator shows damped oscillations with damping time constant $\tau = 2m/b$. Consider such an oscillator where the mass m consists of a square loop of wire (side length L that oscillators with one end in a uniform magnetic field B, as shown in the figure. The wire has cross-sectional area a and resistivity ρ. Compute b in terms of a, ρ, L and B.



Solution. See: `https://tbp.berkeley.edu/exams/424/download/`, problem 6.

Problem 27.6: (Huang, Final Fall 2007) An inductor consists of two indentical conducting planes, separated by a distance d, each carrying a current density $i = I/h$ where I is the total current. Assume that $d \ll h$ and $d \ll w$.

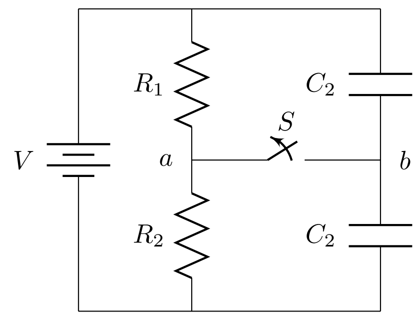
- Derive the magnetic field between the two plates in terms of the current density i.
- Derive the magnetic field outside the two planes in terms of the current density i.
- Find the magnetic flux through the surface of the shaded area (Please specify the direction of the normal for the surface.)
- Calculate the inductance of this inductor.



Solution. See: `https://tbp.berkeley.edu/exams/1586/download/`, problem 5.

Problem 27.7: (Lee, Final Fall 2007)

Two resistors and two uncharged capacitors are arranged as shown in the figure below. Then, a potential difference, V, is applied across the combination as shown.

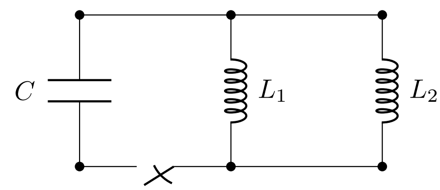


1. What is the potential at point a with S open?
2. What is the potential at point b with S open?
3. After the switch is closed, what is the final potential at point b?
4. What is the total amount of charge that flows through the switch after it is closed?

Solution. See: `https://tbp.berkeley.edu/exams/1628/download/`, problem 2.

Problem 27.8: (Speliotopoulos, Final Fall 2012) Consider the following LC circuit. The capacitor on the left is initially charged with a charge Q_0. If the switch is flipped at $t = 0$, find the following:

1. The shortest time T that it takes for the capactor to fully discharge.
2. The current through the capacitor at time T.



Solution. See: `https://tbp.berkeley.edu/exams/4317/download/`, problem 7. This solution is more complicated than necesary; one should first use the inductor parallel law to find an equivalent circuit with a single inductor.