A Universal Operator Growth Hypothesis

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Acknowledgements

Collaborators

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Advisor

Joel Moore
Quantum Mechanics

Microscopic description of the system.

Example: Chaotic Ising Model

\[ H = \sum_i X_i + 1.05 Z_i Z_{i+1} + 0.59 Z_i \]

Correlation functions:

\[ C(t) = \langle \mathcal{O}(t, x) \mathcal{O}(0) \rangle \]

Hard Solution: Hamiltonian dynamics

\[ \mathcal{O}(t) = e^{-iHt} \mathcal{O} e^{iHt}. \]

Exact and reversible dynamics.

Hydrodynamics

Macroscopic description of quantum systems as classical PDEs.

Example: Diffusion of energy

\[ \frac{\partial}{\partial t} \varepsilon(t, x) = D \nabla^2 \varepsilon(t, x) + \nabla f, \]

with \( D \) diffusion, \( f \) thermal noise.

Easy Solution: Green’s function

\[ G(i\omega, k) = \frac{1}{i\omega + Dq^2} \]

Approximate & irreversible dynamics.
Quantum Mechanics

\[ H = \sum_i X_i + 1.05 Z_i Z_{i+1} + 0.59 Z_i \]

\[ \frac{\partial}{\partial t} \varepsilon(t, x) = D \nabla^2 \varepsilon(t, x) + \nabla f \]

\[ D = ? \]

Hydrodynamics
$H = \sum_i X_i + 1.05 Z_i Z_{i+1} + 0.59 Z_i$

$\frac{\partial}{\partial t} \epsilon(t, x) = D \nabla^2 \epsilon(t, x) + \nabla f$

$D = ?$

A Universal Operator Growth Hypothesis leads to computable hydrodynamics
The Graph of Operators

**Example:** Chaotic Ising Model

\[ H = \sum \limits_i X_i + 1.05 Z_i Z_{i+1} + 0.5 Z_i. \]

**Problem:** Compute \( C(t) = \langle O(t)O(0) \rangle. \)

\[ O(t) = e^{-iHt}Oe^{iHt} \]

\[ = O - it[H,O] + (-it)^2[H,[H,O]] + \cdots \]
The Graph of Operators

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Let's compute!

\[
O = X_1
\]
The Graph of Operators

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\[ H = \sum_i X_i + 1.05 Z_i Z_{i+1} + 0.5 Z_i. \]

Problem: Compute \( C(t) = \langle \mathcal{O}(t) \mathcal{O}(0) \rangle. \)

\[ \mathcal{O}(t) = e^{-iHt} \mathcal{O} e^{iHt} \]
\[ = \mathcal{O} - it[H, \mathcal{O}] + (-it)^2[H, [H, \mathcal{O}]] + \cdots \]

Let’s compute!

\[ \mathcal{O} = X_1 \]
\[ [H, \mathcal{O}] = 1.05 i Y_1 Z_2 + 1.05 i Z_1 Y_2 + 0.5 i Y_1 \]
The Graph of Operators

**Example:** Chaotic Ising Model

\[ H = \sum_i X_i + 1.05Z_iZ_{i+1} + 0.5Z_i. \]

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O(t) = e^{-iHt}Oe^{iHt} = O - it[H,O] + (-it)^2[H,[H,O]] + \cdots
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Let's compute!

\[ O = X_1 \]

\[ [H, O] = 1.05iY_1Z_2 + 1.05iZ_1 Y_2 + 0.5iY_1 \]

\[ [H, [H, O]] = 2.1Z_1Z_2 - 2.1Y_1 Y_2 \]

\[ + 1.05^2Z_0X_1 Z_2 + 1.05^2X_1 + 1.05^2X_2 + 1.05^2Z_1X_2 Z_3 \]

\[ + 0.55H_1 Z_0 + 0.55Z_1 H + 0.55 H_2 \]
The Graph of Operators

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+ 1.05^2 Z_0 X_1 Z_2 + 1.05^2 X_1 + 1.05^2 X_2 + 1.05^2 Z_1 X_2 Z_3 \\
+ 0.25 i Y_1 Z_2 + 0.25 i Z_1 Y_2
\]
The Graph of Operators

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Recursion Method

**Idea:** There are so many operators they act as a *thermodynamic bath*. Dynamics should be *universal*.
Recursion Method

Idea: There are so many operators they act as a thermodynamic bath. Dynamics should be universal.

Recursion Method: Restate problem as 1d tight-binding model.

\[
C(t) = \langle \mathcal{O}(t) \mathcal{O}(0) \rangle = \varphi_0(t)
\]

\[
i \partial_t \varphi_n = L_{nm} \varphi_m
\]

\[
L_{nm} = \begin{pmatrix}
0 & b_1 & 0 & 0 & \cdots \\
b_1 & 0 & b_2 & 0 & \cdots \\
0 & b_2 & 0 & b_3 & \cdots \\
0 & 0 & b_3 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

The \( b_n \)'s are called Lanczos Coefficients.
Krylov Vectors

- The Liouvillian is \( \mathcal{L} := [H, \cdot] \), and \( \mathcal{O}(t) = e^{i\mathcal{L}t}\mathcal{O}(0) \).
- Take the sequence \( \{\mathcal{O}, \mathcal{L}\mathcal{O}, \mathcal{L}^2\mathcal{O}, \ldots\} \) and apply Gram-Schmidt to produce an orthogonal basis \( \{\mathcal{O}_0 = \mathcal{O}, \mathcal{O}_1, \mathcal{O}_2, \ldots\} \).
- The Liouvillian is tridiagonal in this basis
  \[
  L_{nm} := \text{Tr}[\mathcal{O}_n^\dagger \mathcal{L}\mathcal{O}_m] = \begin{pmatrix}
  0 & b_1 & 0 & 0 & \cdots \\
  b_1 & 0 & b_2 & 0 & \cdots \\
  0 & b_2 & 0 & b_3 & \cdots \\
  0 & 0 & b_3 & 0 & \cdots \\
  \vdots & \vdots & \vdots & \vdots & \ddots 
  \end{pmatrix}.
  \]
- 1D Wavefunction defined by \( \varphi_n(t) := \text{Tr}[^{\dagger}\mathcal{O}_n^\dagger\mathcal{O}(t)] \).
- The \( b_n \)'s are called **Lanczos coefficients** and the \( \mathcal{O}_n \)'s are called **Krylov vectors**.

**Recursion Method**

**Recursion Method:** Restate problem as 1d tight-binding model.

\[ C(t) = \langle O(t)O(0) \rangle = \varphi_0(t) \]

\[ i\partial_t \varphi_n = L_{nm}\varphi_m \]

\[ L_{nm} = \begin{pmatrix} 0 & b_1 & 0 & 0 & \cdots \\ b_1 & 0 & b_2 & 0 & \cdots \\ 0 & b_2 & 0 & b_3 & \cdots \\ 0 & 0 & b_3 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \]

- Exact restatement of \( C(t) \) in terms of **Lanczos coefficients** \( \{b_n\}_{n=1}^{\infty} \).
- Old method, dating back to the 1980s. See:
  - D.C. Mattis. 1981
- We can compute a few dozen \( b_n \)'s through numerics.
- “Classification” of dynamics.

<table>
<thead>
<tr>
<th>Asymptotic</th>
<th>Growth Rate</th>
<th>System Type</th>
</tr>
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<tbody>
<tr>
<td>( b_n \sim O(1) )</td>
<td>constant</td>
<td>Free models</td>
</tr>
<tr>
<td>( b_n \sim O(\sqrt{n}) )</td>
<td>square-root</td>
<td>Integrable models</td>
</tr>
<tr>
<td>( b_n \sim ??? )</td>
<td>???</td>
<td>Chaotic models</td>
</tr>
<tr>
<td>( b_n \geq O(n) )</td>
<td>superlinear</td>
<td>Disallowed</td>
</tr>
</tbody>
</table>
Chaotic Examples

\[ H_1 = \sum_i X_i X_{i+1} + 0.709 Z_i + 0.9045 X_i \]

\[ H_2 = H_1 + \sum_i 0.2 Y_i \]

\[ H_3 = H_1 + \sum_i 0.2 Z_i Z_{i+1} \]

\[ H(h_X) = \sum_i X_i X_{i+1} - 1.05 Z_i + h_X X_i \]

\[ H^{(q)}_{SYK} = i^{q/2} \sum_{1 \leq i_1 < i_2 < \cdots < i_q \leq N} J_{i_1 \cdots i_q} \gamma_{i_1} \cdots \gamma_{i_q}, \]

\[ \bar{J}_{i_1 \cdots i_q}^2 = 0, \]

\[ \bar{J}_{i_1 \cdots i_q}^2 = (q - 1)! \frac{N q - 1}{N^{q-1}} J^2 \]
Hypothesis: In a chaotic quantum system, the Lanczos coefficients $b_n$ are asymptotically linear, i.e. for $\alpha, \gamma \geq 0$,

$$b_n \xrightarrow{n \gg 1} \alpha n + \gamma.$$ 

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<td>Linear</td>
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<tr>
<td>$b_n \gtrsim O(n)$</td>
<td>Superlinear</td>
</tr>
</tbody>
</table>

Free models

Integrable models

Chaotic models

Disallowed
Consider

\[ \tilde{b}_n := \alpha \sqrt{n(n - 1 + \eta)} \quad \text{as} \quad n \gg 1 \rightarrow \alpha n + \gamma. \]

We can solve this exactly:

\[ \tilde{\varphi}_n(t) = \sqrt{\frac{(\eta)_n}{n!}} \tanh(\alpha t)^n \text{sech}(\alpha t)^{\eta} \]

where \((\eta)_n = \eta(\eta + 1) \cdots (\eta + n + 1)\).

Expected “position” in the 1D chain is

\[ (n(t)) = \eta \sinh(\alpha t)^2 \sim e^{2\alpha t}. \]

The wavefunction runs away “irreversibly” into the 1D chain.
Numerical Coefficients

Asymptotic Coefficients: $\tilde{b}_n = \alpha n$

$b_1 \quad \cdots \quad b_N$

$\varphi_0 \quad \varphi_1 \quad \cdots \quad \varphi_{N-1} \quad \varphi_N$

$\tilde{b}_5 \quad \tilde{b}_6 \quad \tilde{b}_7$

$\tilde{b}_n = \alpha n$

$\tilde{G}(N)(z) = \Gamma(N+1)\Gamma(z+1)\left\{ \times 2F_1\left(\frac{N+1}{2},z+1\right)\frac{N+3}{2}\right\}^{-1}$
Numerical Coefficients

Asymptotic Coefficients: $\tilde{b}_n = \alpha n$

\[
G(z) = \int dt \ e^{izt} \langle O(t)O(0) \rangle
\approx \frac{1}{z - \frac{b_1^2}{b_1}} \frac{1}{z - \frac{b_2^2}{b_2}} \cdots \frac{1}{z - \frac{b_N^2}{b_N}} \quad \text{G}(\tilde{\text{G}})(z)
\]

\[
\tilde{\text{G}}(\tilde{\text{G}})(z) = \Gamma(N + 1) \Gamma \left( \frac{z + 1}{2} \right) \times _2F_1 \left( N + 1, \frac{z + 1}{2}, \frac{z + 2N + 3}{2}; -1 \right)
\]
Algorithm

0. Choose a local operator $\mathcal{O}$ whose correlation $C(t) = \text{Tr}[\mathcal{O}(t)\mathcal{O}(0)]$ should be hydrodynamical.

1. Compute $b_1, \ldots, b_N$ via infinite exact diagonalization and fit the slope $\alpha$.

2. Stitch together the $b_n$’s and the asymptotic solution $\tilde{G}(N)$.

3. Identify the pole closest to the origin to extract the hydrodynamical dispersion relation.
Diffusion in the Chaotic Ising Model

Chaotic Ising Model

\[ H = \sum_j X_j + 1.05Z_jZ_{j+1} + 0.5Z_j \]

Initial operator at wavevector \( k \):

\[ \mathcal{O}_k = \sum_j e^{ikj} (X_j + 1.05Z_jZ_{j+1} + 0.5Z_j) \]

We see the dispersion relation for diffusion

\[
\frac{d}{dt} \epsilon(t, x) = D \nabla^2 \epsilon(t, x).
\]
Summary

- The hypothesis governs operator growth in chaotic, closed quantum systems
  \[ b_n \xrightarrow{n\gg 1} \alpha n + \gamma. \]

- Emergence of hydrodynamics in a computationally tractable scheme.

- The **operator growth rate** \( \alpha \) also controls the growth of complexity and chaos in quantum systems. [Talk in Session P24 by Xiangyu Cao]

<table>
<thead>
<tr>
<th>SYK-( q )</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>7</th>
<th>10</th>
<th>( \infty )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha/\mathcal{J} )</td>
<td>0</td>
<td>0.461</td>
<td>0.623</td>
<td>0.800</td>
<td>0.863</td>
<td>1</td>
</tr>
<tr>
<td>( \lambda_L/(2\mathcal{J}) )</td>
<td>0</td>
<td>0.454</td>
<td>0.620</td>
<td>0.799</td>
<td>0.863</td>
<td>1</td>
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- arXiv:1812.08657
Chaos & Complexity

Operator Growth Hypothesis

$\mathbf{b_n \sim \alpha n}$

Hydro

Finite Temperature

Thermalization

Finite Temperature
Other guises of $\alpha$: relation to the spectral function, analytic structure of $C(t)$, experimental probes.

The exponential growth of Krylov complexity suggests that $2\alpha$ can be interpreted as a Lyapunov exponent.

Formal of complexity: Krylov-complexity and Q-complexities

Theorem: Krylov complexity grows faster than any other complexity, including operator size and OTOCs.

The theorem above implies the so-called “quantum bound on chaos” at low temperatures.

Most of this story carries over directly to the classical case.

One can show the SYK model obeys the hypothesis directly, and compute most of these quantities exactly.
Extra Slide: Operator Space

We move up a level of abstraction from the space of states to the **space of operators**.

- Operators $\mathcal{O}$ are now “kets”, $|\mathcal{O}\rangle$.
- e.g. $|\mathcal{O}\rangle = X_1 \otimes Y_2 \otimes Z_3 + 0.3 Y_1 \otimes X_2$.
- A basis of operators is the set of **Pauli Strings**
  
  $$|\alpha\rangle = \sigma^{\alpha_1} \otimes \sigma^{\alpha_2} \otimes \cdots \otimes \sigma^{\alpha_n}$$

  for $\alpha_i = 0, 1, 2, 3$.
- Operator inner product:

  $$(A|B) := \text{Tr}[A^\dagger B].$$

- The **Liouvillian superoperator** gives the commutator of an operator against the Hamiltonian
  
  $$\mathcal{L} = [H, \cdot].$$

- Heisenburg equation of motion

  $$-i \frac{d\mathcal{O}}{dt} = [H, \mathcal{O}] \rightarrow -i \frac{d|\mathcal{O}\rangle}{dt} = \mathcal{L} |\mathcal{O}\rangle.$$ 

- By Baker-Campbell-Hausdorff,

  $$\mathcal{O}(t) = e^{iHt} \mathcal{O} e^{-iHt} \rightarrow |\mathcal{O}(t)\rangle = e^{-i\mathcal{L}t} |\mathcal{O}\rangle.$$ 

- Operators evolve in operator space like states in state space.
Extra Slide: The Recursion Method

\[ \tilde{G}(z) = \sum_{\text{paths}} \tilde{b}_1 \tilde{b}_2 \tilde{b}_3 \tilde{b}_4 \tilde{b}_5 \tilde{b}_6 \tilde{b}_7 \]

\[ \tilde{G}'(z) = \sum_{\text{paths}} \]

\[ = \]

\[ + \sum_{\text{paths}} \]

\[ = \frac{1}{1 + \tilde{b}_1^2 \tilde{G}^{(1)}(z)} \]

\[ = \frac{1}{1 + \frac{\tilde{b}_1^2}{1 + \frac{\tilde{b}_2^2}{1 + \frac{\tilde{b}_3^2 \tilde{G}^{(3)}(z)}{1 + \tilde{b}_2^2 \tilde{G}^{(3)}(z)}}}} \]