Acknowledgements

Collaborators

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Advisor

Joel Moore
Quantum Mechanics

Microscopic description of the system. **Example:** Chaotic Ising Model

\[ H = \sum_i X_i + 1.05 Z_i Z_{i+1} + 0.59 Z_i \]

**Correlation functions:**

\[ C(t) = \langle O(t, x) O(0) \rangle \]

**Hard Solution:** Hamiltonian dynamics

\[ O(t) = e^{-iHt} O e^{iHt}. \]

Exact and *reversible* dynamics.

Hydrodynamics

Macroscopic description of quantum systems as classical PDEs. **Example:** Diffusion of energy

\[ \frac{\partial}{\partial t} \varepsilon(t, x) = D \nabla^2 \varepsilon(t, x) + \nabla f, \]

with \( D \) diffusion, \( f \) thermal noise. **Easy Solution:** Green’s function

\[ G(i\omega, k) = \frac{1}{i\omega + Dq^2} \]

Approximate & *irreversible* dynamics.
Operator Growth

or

How I learned to stop worrying and love tridiagonalization.
Operator Space

Inspiration: random unitary circuits.


Consider a spin-1/2 system in $d$-dimensions with translation invariance.

$$H = \sum_{x \in \mathbb{Z}^d} h_x.$$

We abstract to the space of operators.

- operators are “rounded” kets $|\mathcal{O}\rangle$
- an example is $|\mathcal{O}\rangle = X_1 \otimes Y_2 \otimes Z_3 + 0.3 Y_1 \otimes X_2$
- the inner product is $(A|B) := \text{Tr}[A^\dagger B]/\text{Tr}[1]$
- the Liouvillian generalizes the Hamiltonian $\mathcal{L} = [H, \cdot]$.
- time-evolution from Heisenberg EOM $-i \frac{d|\mathcal{O}\rangle}{dt} = \mathcal{L} |\mathcal{O}\rangle$.
- solution $|\mathcal{O}(t)\rangle = e^{i\mathcal{L}t} |\mathcal{O}\rangle$. 
Three Observables

A. Correlation Function

\[ C(t) := (\mathcal{O}(t) | \mathcal{O}(0)) = \sum_{n=0}^{\infty} \frac{\mu_{2n}}{(2n)!} (it)^{2n} \quad \text{with moments} \quad \mu_{2n} = (\mathcal{O} | \mathcal{L}^{2n} \mathcal{O}) . \]

B. Green’s Function

\[ G(z) := (\mathcal{O} | \frac{1}{z - \mathcal{L}} | \mathcal{O}) = i \int_0^{\infty} e^{-izt} C(t) \, dt = \sum_{n=0}^{\infty} \frac{\mu_{2n}}{z^{2n+1}} . \]

C. Spectral Function

\[ \Phi(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} C(t) \, dt = \sum_{E,E'} |\langle E | \mathcal{O} | E' \rangle|^2 \delta(\omega - (E - E')) . \]
The Graph of Operators

Example: Chaotic Ising Model

\[ H = \sum_i X_i + 1.05 Z_i Z_{i+1} + 0.5 Z_i. \]

Problem: Compute \( C(t) = \langle \mathcal{O} | e^{i \mathcal{L} t} | \mathcal{O} \rangle. \)

\[ \mathcal{O}(t) = e^{i \mathcal{L} t} \mathcal{O} = \mathcal{O} + (it) \mathcal{L} \mathcal{O} + (it)^2 \mathcal{L}^2 \mathcal{O} + \cdots \]
The Graph of Operators

**Example:** Chaotic Ising Model

\[ H = \sum_i X_i + 1.05 Z_i Z_{i+1} + 0.5 Z_i. \]

**Problem:** Compute \( C(t) = (O | e^{iL_t} | O). \)

\[ O(t) = e^{iL_t} O = O + (it)L O + (it)^2 L^2 O + \cdots \]

Let’s compute!

\( O = X_1 \)
The Graph of Operators

**Example:** Chaotic Ising Model

\[ H = \sum_i X_i + 1.05 Z_i Z_{i+1} + 0.5 Z_i. \]

**Problem:** Compute \( C(t) = \langle \mathcal{O} | e^{i \mathcal{L} t} | \mathcal{O} \rangle \).

\[ \mathcal{O}(t) = e^{i \mathcal{L} t} \mathcal{O} = \mathcal{O} + (it) \mathcal{L} \mathcal{O} + (it)^2 \mathcal{L}^2 \mathcal{O} + \cdots \]

Let’s compute!

\[ \mathcal{O} = X_1 \]

\[ \mathcal{L} \mathcal{O} = 1.05 i Y_1 Z_2 + 1.05 i Z_1 Y_2 + 0.5 i Y_1 \]
The Graph of Operators

**Example:** Chaotic Ising Model

\[ H = \sum_i X_i + 1.05 Z_i Z_{i+1} + 0.5 Z_i. \]

**Problem:** Compute \( C(t) = (\mathcal{O}|e^{i\mathcal{L}t}|\mathcal{O}). \)

\[ \mathcal{O}(t) = e^{i\mathcal{L}t} \mathcal{O} = \mathcal{O} + (it)\mathcal{L}\mathcal{O} + (it)^2 \mathcal{L}^2 \mathcal{O} + \cdots \]

Let’s compute!

\[ \mathcal{O} = X_1 \]

\[ \mathcal{L}\mathcal{O} = 1.05i Y_1 Z_2 + 1.05i Z_1 Y_2 + 0.5i Y_1 \]

\[ \mathcal{L}^2 \mathcal{O} = 2.1Z_1 Z_2 - 2.1 Y_1 Y_2 \]

\[ + 1.05^2 Z_0 X_1 Z_2 + 1.05^2 X_1 + 1.05^2 X_2 + 1.05^2 Z_1 X_2 Z_3 \]

\[ + 0.525 X_1 Z_2 + 0.525 Z_1 X_2 + 0.25 X_1. \]
The Graph of Operators

Example: Chaotic Ising Model

\[ H = \sum_{i} X_i + 1.05 Z_i Z_{i+1} + 0.5 Z_i. \]

Problem: Compute \( C(t) = (\mathcal{O}|e^{i\mathcal{L}t}|\mathcal{O}). \)

\[ \mathcal{O}(t) = e^{i\mathcal{L}t} \mathcal{O} = \mathcal{O} + (it)\mathcal{L}\mathcal{O} + (it)^2\mathcal{L}^2\mathcal{O} + \cdots \]

Let's compute!

\[ \mathcal{O} = X_1 \]
\[ \mathcal{L}\mathcal{O} = 1.05 i Y_1 Z_2 + 1.05 i Z_1 Y_2 + 0.5 i Y_1 \]
\[ \mathcal{L}^2\mathcal{O} = 2.1 Z_1 Z_2 - 2.1 Y_1 Y_2 + 1.05^2 Z_0 X_1 Z_2 + 1.05^2 X_1 + 1.05^2 X_2 + 1.05^2 Z_1 X_2 Z_3 + 0.525 X_1 Z_2 + 0.525 Z_1 X_2 + 0.25 X_1. \]
The Graph of Operators

Example: Chaotic Ising Model

\[ H = \sum_i X_i + 1.05 Z_i Z_{i+1} + 0.5 Z_i. \]

Problem: Compute \( C(t) = (\mathcal{O}|e^{i\mathcal{L}t}|\mathcal{O}). \)

\[ \mathcal{O}(t) = e^{i\mathcal{L}t} \mathcal{O} = \mathcal{O} + (it)\mathcal{L}\mathcal{O} + (it)^2 \mathcal{L}^2 \mathcal{O} + \cdots \]

Let’s compute!

\[ \mathcal{O} = X_1 \]

\[ \mathcal{L}\mathcal{O} = 1.05iY_1Z_2 + 1.05iZ_1Y_2 + 0.5iY_1 \]

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\[ + 1.05^2 Z_0X_1Z_2 + 1.05^2 X_1 + 1.05^2 X_2 + 1.05^2 Z_1X_2Z_3 \]

\[ + 0.525X_1Z_2 + 0.525Z_1X_2 + 0.25X_1. \]
The Basic Idea

- Operators flow from simple to complex, eventually becoming too complex to compute.
- Complex operators are superpositions of a thermodynamically large number of Pauli strings.
- A sufficiently complex operator should admit a universal description.
- Our goal now is to formulate this universal description.
The Lanczos Algorithm

- Take the sequence \( \{\mathcal{O}, \mathcal{L}\mathcal{O}, \mathcal{L}^2\mathcal{O}, \ldots\} \) and apply Gram-Schmidt to orthogonalize \( \{\mathcal{O}_0, \mathcal{O}_1, \mathcal{O}_2, \ldots\} \).
- Explicitly, \( |\mathcal{O}_1\rangle := b_1^{-1} \mathcal{L} |\mathcal{O}_0\rangle \), \( b_1 := (\mathcal{O}_0 \mathcal{L} |\mathcal{L}\mathcal{O}_0\rangle)^{1/2} \),
  \( |A_n\rangle := \mathcal{L} |\mathcal{O}_{n-1}\rangle - b_{n-1} |\mathcal{O}_{n-2}\rangle \),
  \( b_n := (A_n |A_n\rangle)^{1/2} \) \textbf{“Lanczos Coefficients”}
  \( |\mathcal{O}_n\rangle := b_n^{-1} |A_n\rangle \) \textbf{“Krylov vectors”}
- The Liouvillian is tridiagonal in this basis:

\[
L_{nm} := (\mathcal{O}_n^\dagger |\mathcal{L}|\mathcal{O}_m) = \\
\begin{pmatrix}
0 & b_1 & 0 & 0 & \cdots \\
b_1 & 0 & b_2 & 0 & \cdots \\
0 & b_2 & 0 & b_3 & \cdots \\
0 & 0 & b_3 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

The Recursion Method

Define the 1D wavefunction by

$$|\mathcal{O}(t)\rangle = \sum_{n=0}^{\infty} \varphi_n(t) |\mathcal{O}_n\rangle, \quad \varphi_n(t) := (\mathcal{O}_n |\mathcal{O}(t)\rangle).$$

The operator evolves as $-i \frac{d}{dt} \mathcal{O} = \mathcal{L} \mathcal{O}$, and $\mathcal{L}$ is tridiagonal:

$$-i \partial_t \varphi_n = b_{n+1} \varphi_{n+1} + b_n \varphi_{n-1}, \quad \varphi_n(0) = \delta_{n0}.$$ 

The autocorrelation is just the wavefunction on site zero:

$$C(t) = (\mathcal{O}_0 |\mathcal{O}(t)\rangle) = \varphi_0(t).$$

This is called the **recursion method** and dates back to the 1980s.

Encodings of Dynamics

A. Correlation Function
\[ C(t) := (\mathcal{O}|e^{i\mathcal{L}t}|\mathcal{O}) = \sum_{n=0}^{\infty} \frac{\mu_{2n}}{(2n)!}(it)^{2n} \]

B. Green’s Function
\[ G(z) := (\mathcal{O}\left| \frac{1}{z - \mathcal{L}} \right|\mathcal{O}) = \sum_{n=0}^{\infty} \frac{\mu_{2n}}{z^{2n+1}} \]

C. Spectral Function
\[ \Phi(\omega) := \sum_{E, E'} |\langle E|\mathcal{O}|E'\rangle|^2 \delta(\omega - (E - E')) \]
Encodings of Dynamics

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C. Spectral Function

\[ \Phi(\omega) := \sum_{E,E'} |\langle E|\mathcal{O}|E'\rangle|^2 \delta(\omega - (E - E')) \]

D. Lanczos Coefficients

\[ \{b_n\}_{n=1}^{\infty} \quad \& \quad -i\partial_t \varphi_n = b_{n+1}\varphi_{n+1} + b_n\varphi_{n-1} \]
Empirical Patterns of Dynamics

<table>
<thead>
<tr>
<th>Asymptotic</th>
<th>Growth Rate</th>
<th>System Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b_n \sim O(1)$</td>
<td>constant</td>
<td>Free models</td>
</tr>
<tr>
<td>$b_n \sim O(\sqrt{n})$</td>
<td>square-root</td>
<td>Integrable models</td>
</tr>
<tr>
<td>$b_n \sim ???$</td>
<td>???</td>
<td>Chaotic models</td>
</tr>
<tr>
<td>$b_n \gtrsim O(n)$</td>
<td>superlinear</td>
<td>Disallowed</td>
</tr>
</tbody>
</table>

Chaotic Examples

\[ H_1 = \sum_i X_i X_{i+1} + 0.709 Z_i + 0.9045 X_i \]

\[ H_2 = H_1 + \sum_i 0.2 Y_i \]

\[ H_3 = H_1 + \sum_i 0.2 Z_i Z_{i+1} \]

\[ H^{(q)}_{\text{SYK}} = j^{q/2} \sum_{1 \leq i_1 < i_2 < \cdots < i_q \leq N} J_{i_1 \cdots i_q} \gamma_{i_1} \cdots \gamma_{i_q}, \]

\[ \overline{J_{i_1 \cdots i_q}^2} = 0, \]

\[ \overline{J_{i_1 \cdots i_q}^2} = \frac{(q - 1)!}{N^{q-1}} j^2 \]
**Hypothesis:** In a chaotic\(^1\) quantum system, the Lanczos coefficients \(b_n\) are asymptotically linear, i.e. for \(\alpha, \gamma \geq 0\),

\[
b_n \xrightarrow{n \gg 1} \alpha n + \gamma.
\]

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</tr>
<tr>
<td>(b_n \approx O(n))</td>
<td>Superlinear</td>
<td>Disallowed</td>
</tr>
</tbody>
</table>

\(^1\)i.e. non-integrable
Hypothesis: In a chaotic quantum system, the Lanczos coefficients have asymptotics

\[ b_n = \begin{cases} 
A \frac{n}{W(n)} + O(1) \sim \frac{A_n}{\log n} & \text{if } d = 1 \\
\alpha n + \gamma + o(1) \sim \alpha n & \text{if } d \geq 2 \end{cases} \]

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</table>

1D correction via Alex Avdoshkin & thesis of G.D. Bouch
Thermalization

The Hypothesis enforces “irreversible” dynamics.
Exact Asymptotic Behavior

Model

\[ \tilde{b}_n := \alpha \sqrt{n(n - 1 + \eta)} \xrightarrow{n \gg 1} \alpha n + \gamma. \]

Exact solution

\[ \tilde{\varphi}_n(t) = \sqrt{\frac{(\eta)_n}{n!}} \tanh(\alpha t)^n \sech(\alpha t)^\eta \]

where \((\eta)_n = \eta(\eta + 1) \cdots (\eta + n + 1)\).

Define the Krylov space position operator

\[ (n(t)) := \sum_{n \in \mathbb{N}} n |\varphi_n(t)|^2 = \eta \sinh(\alpha t)^2 \sim e^{2\alpha t} \]

The wavefunction runs away "irreversibly" into the 1D chain.
Four Observables

**D. Lanczos Coefficients**  \( b_n \sim \alpha n \)
Four Observables

D. Lanczos Coefficients $b_n \sim \alpha n$

\begin{align*}
\Phi(\omega) &\sim e^{-\frac{\pi|\omega|}{2\alpha}} \\
C(t) &\text{ analytic}
\end{align*}
Four Observables

D. Lanczos Coefficients \( b_n \sim \alpha n \)

B. Green’s Function

\[ G(z) = \frac{1}{z - b_1^2} \]
\[ z - \frac{b_2^2}{z} \]
\[ z - \frac{b_3^2}{z} \]

C. Spectral Function

\[ \Phi(\omega) \sim e^{-\frac{\pi |\omega|}{2\alpha}} \]
Four Observables

**D.** Lanczos Coefficients \( b_n \sim \alpha n \)

**B.** Green’s Function
\[
G(z) = \frac{1}{z - \frac{b_1^2}{2} z - \frac{b_2^2}{2} z - \frac{b_3^2}{2} \ldots}
\]

**C.** Spectral Function
\[\Phi(\omega) \sim e^{-\frac{\pi|\omega|}{2\alpha}}\]

**A.** Correlation Function
\[C(t) \text{ analytic}\]
Quantum Chaos

The Lanczos coefficients “diagnose” quantum chaos.
Hallmarks of Quantum Chaos

- **Level Statistics** give a “highly microscopic” indicator of chaos. With $\delta E_n := E_{n+1} - E_n$, the r-statistic is:

$$r = \frac{\min \{\delta E_n, \delta E_{n+1}\}}{\max \{\delta E_n, \delta E_{n+1}\}} \approx \begin{cases} 0.386 & \text{integrable} \\ 0.530 & \text{chaotic} \end{cases}$$

- **Eigenstate Thermalization Hypothesis** predicts matrix elements of chaotic systems

$$\langle E_n | O | E_m \rangle = O(E)\delta_{nm} + e^{-S(E)} f_\omega(E, \omega) R_{nm}$$

| Dynamics   | Level Stats. | $b_n$   | $\Phi(\omega)$ or $|f_\omega|^2$ |
|------------|--------------|---------|----------------------------------|
| Free       | -            | $O(1)$  | $\theta(|\omega - 2W|)$          |
| Integrable | Poisson      | $O(\sqrt{n})$ | $O(e^{-\omega^2})$ |
| Chaotic    | GOE          | $O(n)$  | $O(e^{-\omega})$                 |

- **Exponential sensitivity**
Example 1: Chaotic Ising Model

Model

\[ H = \sum_i J [X_i X_{i+1} + hZ_i] + h_X X_i \]

Dynamics In the thermodynamic limit,

\[ \begin{cases} h_X = 0 & \text{free model} \\ h_X > 0 & \text{chaotic model} \end{cases} \]

Perturbation Theory Resonances appear at order \( O(h_X/J) \).

\[ b_n \approx \begin{cases} O(1) & n < O(h_X/J) \\ O(n) & n > O(h_X/J) \end{cases} \]
Example 2: SYK Model

Model\(^1\)

\[ H = tH_{\text{SYK}}^{(2)} + UH_{\text{SYK}}^{(4)} \]

\[ H_{\text{SYK}}^{(q)} = \frac{i^q}{2} \sum_{1 \leq i_1 < i_2 < \ldots < i_q \leq N} J_{i_1 \ldots i_q} \gamma_{i_1} \cdots \gamma_{i_q}, \]

\[ \overline{J_{i_1 \ldots i_q}^2} = 0, \quad \overline{J_{i_1 \ldots i_q}^2} = \frac{(q - 1)!}{N^{q-1}} J^2 \]

Dynamics

\[ q = \begin{cases} 2 & \text{free model} \\ 4, 6, 8, \ldots & \text{chaotic model} \end{cases} \]

\(^1\)Sachdev, Ye, 1993; Parcollet, Georges, 1999; Kitaev, 2015; Maldacena, Stanford, 2016, etc.
Example 3: XXZ+NHN

Model

\[ H_1 = \sum_i X_i X_{i+1} + Y_i Y_{i+1} + \Delta Z_i Z_{i+1} \]

\[ + \lambda \left( \sum_i X_i X_{i+2} + Y_i Y_{i+2} + \Delta Z_i Z_{i+2} \right) \]

\[ A := \sum_i Z_i Z_{i+1}, B := \sum_i S_i^+ S_{i+2}^- + \text{h.c.} \]

Dynamics

\[
\begin{cases}
\Delta = 0 \& \lambda = 0 & \text{free model} \\
\Delta \neq 0 \& \lambda = 0 & \text{integrable model} \\
\Delta \neq 0 \& \lambda \neq 0 & \text{chaotic model}
\end{cases}
\]

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<td>Ising</td>
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<td>$O(1)$</td>
<td>Analytic</td>
<td>Viswanath &amp; Müller</td>
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</tr>
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<td>SYK$^{(2)}$</td>
<td>$\gamma$</td>
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<td>Analytic</td>
<td>Maldacena, Shenkar, Stanford, 2016</td>
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<tr>
<td>XX</td>
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<td>$O(\sqrt{n})$</td>
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<td>Viswanath &amp; Müller</td>
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<tr>
<td>Free Fermions in Disguise</td>
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<td>see Fendley, 2019.</td>
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<td>MBL</td>
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<tr>
<td>XXZ + NNN</td>
<td>$\hat{ZZ}$</td>
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<td>$O(n)$</td>
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<td>LeBlond, Mallayya, Vidmar, Rigol, 2019.</td>
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<td>SYK$^{(4)}$</td>
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<td>SYK$^{(\infty)}$</td>
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<td>$O(n)$</td>
<td>Analytic</td>
<td>Roberts, Stanford, Streicher, 2018.</td>
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<td>SYK Hopping</td>
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<td>2D Fermi Hubbard</td>
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<td>$O(n)$</td>
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<td>Huang, private comm.</td>
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<td>Bouch Model</td>
<td>$\hat{X}$</td>
<td>Chaotic</td>
<td>$O(n)$</td>
<td>Analytic</td>
<td>Bouch, 2015</td>
</tr>
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</table>
Complexity

The Lanczos coefficients quantify quantum chaos.
I will introduce a measure of quantum chaos ("K-Complexity") that is
1. easy to interpret
2. easy to compute
3. works in all quantum systems (not semiclassical).
Exponential Sensitivity

- A hallmark of chaos is *exponential sensitivity* to small perturbations.
- Classically, this is measured by the Lyapunov exponent.

\[ \text{Out-of-time-order commutator} \ (t) := \langle [O(t), V] | [O(t), V] \rangle \sim e^{\lambda_L t} \]

\( \lambda_L \) is called the quantum Lyapunov exponent.

- High-energy theorists have shown a “universal bound on chaos”: for \( T \to 0 \),
  \[ 2 \lambda_L \leq 2 \pi T. \] (1)

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$$\text{OTOC}(t) := ([\mathcal{O}(t), V][\mathcal{O}(t), V]) \sim e^{\lambda_L t}.$$ 

and $\lambda_L$ is called the \textbf{quantum Lyapunov exponent}.\footnote{Maldacena, Shenker, Stanford, 2016.}
Exponential Sensitivity

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- Classically, this is measured by the Lyapunov exponent.
- The **Out-of-time-order commutator** generalizes the Lyapunov exponent $\lambda_L$ to semi-classical systems:\n
  $$\text{OTOC}(t) \equiv ([\mathcal{O}(t), V][\mathcal{O}(t), V]) \sim e^{\lambda_L t}.$$\n
  and $\lambda_L$ is called the **quantum Lyapunov exponent**.
- High-energy theorists have shown a “universal bound on chaos”: for $T \to 0$:\n
  $$\lambda_L \leq 2\pi T. \quad (1)$$

---

Out-of-time-order Confusion

Semiclassical OTOCs usually saturate at

\[ t = \begin{cases} 
O(\log(1/\hbar)) & \text{semiclassics} \\
O(\log N) & \text{large-}N \\
O(1) & \text{no small parameter}
\end{cases} \]

Regularization Dependent Choice of norm:

\[ (A|B)_\beta := \text{Tr}[\rho A^\dagger B] \quad \text{“Physical”} \]
\[ (A|B)_W^\beta := \text{Tr}[\rho^{1/2} A^\dagger \rho^{1/2} B] \quad \text{“Wightman”} \]

where \( \rho = e^{-\beta H}/Z \).

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Out-of-time-order Confusion

Semiclassical OTOCs usually saturate at

\[ t = \begin{cases} 
O(\log(1/\hbar)) & \text{semiclassics} \\
O(\log N) & \text{large-}N \\
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\]

where \( \rho = e^{-\beta H}/Z \).

Therefore:

- Difficult to define \( \lambda_L \) in spin chains.
- Must use clever tricks like velocity-dependent Lyapunov exponents.\(^2\)

The Krylov vectors $O_n$ grow successively larger, have more components, need more resources... they are more complex.

Therefore define the **K-Complexity** as

$$(n(t)) := \sum_{n \in \mathbb{N}} n |\varphi_n(t)|^2 \sim e^{2\alpha t}$$

where $\varphi_n(t) := (O_n|O(t))$.

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K-Complexity

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Rigorous Bounds

Proposition: Suppose $T = \infty$. For any local operator, $\exists C > 0$ such that

$$\text{OTOC}(t) \leq C \cdot (n(t)).$$
Rigorous Bounds

**Proposition:** Suppose $T = \infty$. For any local operator, $\exists C > 0$ such that

$$\text{OTOC}(t) \leq C \cdot (n(t)).$$

**Corollary:** Suppose $b_n \asymp \alpha n$ and $T = \infty$. Then, if the quantum Lyapunov exponent $\lambda_L$ is defined,

$$\lambda_L \leq 2\alpha.$$

**Proposition:** Suppose $b_n \asymp \alpha_W n$ using the Wightman regularization. Then

$$2\alpha \leq 2\pi T.$$

**Conjecture:** Under the same assumptions,

$$\lambda_L \leq 2\alpha \leq 2\pi T.$$
The Krylov operators $O_n$ hit the edge at $n = O(L)$ and stop growing.

Random matrix theory then kicks in and orthogonality causes the $b_n$'s to decrease.

The $K$-complexity keeps growing linearly until $t = O(e^L/L)$.

This seems to match quantitatively with the “switchback effect” considered in black hole complexity.\textsuperscript{1,2}

\textsuperscript{1}Barbón, Rabinovici, Shir, Sinha, 2019. \textsuperscript{2}DP et al, in progress.
Chaos and Complexity

\[ (n(t)) := \sum_{n \in \mathbb{N}} n |\varphi_n(t)|^2 \asymp e^{2\alpha t} \]

| Dynamics   | Level Stats. | \( b_n \) | \( \Phi(\omega) \) or \( |f_0|^2 \) | K-Complexity |
|------------|--------------|----------|-----------------|--------------|
| Free       | -            | \( O(1) \) | \( \theta(|\omega - 2W|) \) | \( (n)_t \asymp t \) |
| Integrable | Poisson      | \( O(\sqrt{n}) \) | \( O(e^{-\omega^2}) \) | \( (n)_t \asymp t^2 \) |
| Chaotic    | GOE          | \( O(n) \) | \( O(e^{-\pi\omega/2\alpha}) \) | \( (n)_t \asymp e^{2\alpha t} \) |

Therefore \( \alpha \) is the complexity growth rate.

This is experimentally observable from high-frequency heating.

The complexity is non-perturbative data needed to compute hydrodynamics.
Hydrodynamics

K-complexity gives rise to emergent hydrodynamics.
Numerical Coefficients

$\varphi_0 \varphi_1 \varphi_{N-1} \varphi_N$

$\phi_1 \phi_N$

Asymptotic Coefficients: $\tilde{b}_n \sim \alpha n$

$\tilde{b}_{N+1} \tilde{b}_{N+2} \tilde{b}_{N+3}$

$L = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$

$\tilde{G}(N)(z) \approx \frac{1}{z} z^{-b_2} z^{-b_2^2} \cdots z^{-b_2^N} \tilde{G}_N(z) = \Gamma(N+1) \Gamma(z+1)^2 \times 2 F_1(N+1, z+1, z+2, -1)$
Numerical Coefficients

Asymptotic Coefficients: $\tilde{b}_n \sim \alpha n$

$$G(z) = \int dt \, e^{izt} \langle O(t) O(0) \rangle$$

$$\approx \frac{1}{z - \frac{b_1^2}{b_1}} \frac{1}{z - \frac{b_2^2}{b_2}} \cdots \frac{1}{z - \frac{b_N^2}{b_N} \widetilde{G}^{(N)}(z)}$$

$$\widetilde{G}^{(N)}(z) = \Gamma(N + 1) \Gamma \left( \frac{z + 1}{2} \right) \times _2F_1 \left( N + 1, \frac{z + 1}{2}, \frac{z + 2N + 3}{2}; -1 \right)$$
Algorithm

0. Choose a local operator $\mathcal{O}$ whose correlation $C(t) = \text{Tr}[\mathcal{O}(t)\mathcal{O}(0)]$ should be hydrodynamical.

1. Compute $b_1, \ldots, b_N$ via infinite exact diagonalization and fit the slope $\alpha$.

2. Stitch together the $b_n$'s and the asymptotic solution $\tilde{G}(N)$.

3. Identify the pole closest to the origin to extract the hydrodynamical dispersion relation.
Diffusion in the Chaotic Ising Model

Chaotic Ising Model

\[ H = \sum_j X_j + 1.05 Z_j Z_{j+1} + 0.5 Z_j \]

Initial operator at wavevector \( k \):

\[ O_k = \sum_j e^{ikj} (X_j + 1.05 Z_j Z_{j+1} + 0.5 Z_j) \]

We see the dispersion relation for diffusion

\[ \frac{d}{dt} \epsilon(t, x) = D \nabla^2 \epsilon(t, x). \]
Chaotic Ising Model

\[ H = \sum_j X_j + 1.05 Z_j Z_{j+1} + h_x Z_j \]

As \( h_x \to 0 \), a Drude peak (\( D \to \infty \)) emerges.

*This is a practical method for computing hydrodynamics at strong coupling.*
Finite Temperature

At $T < \infty$ there is a *physical* choice of inner product. Suppose $g$ is an even measure on the thermal circle:

1. $g : [0, \beta] \to \mathbb{R}$ (or a distribution)
2. $g(\lambda) = g(\beta - \lambda)$
3. $\int_0^\beta g(\lambda) = 1$.

Then, with $y := e^{-H}$, there is a $g$-inner product:

$$
(A|B)^g_\beta := \frac{1}{Z(\beta)} \int_0^\beta g(\lambda) \operatorname{Tr}[y^{\beta-\lambda}A^\dagger y^\lambda B] \, d\lambda
$$

Two common choices are:

**Physical** $(A|B)^P_\beta := Z^{-1} \operatorname{Tr}[\rho A^\dagger B] + (A \leftrightarrow B)$

**Wightman** $(A|B)^W_\beta := Z^{-1} \operatorname{Tr}[\rho^{1/2}A^\dagger \rho^{1/2}B] + (A \leftrightarrow B)$

In the limit $T \to \infty$ or $\beta \to 0$, these all give the Hilbert-Schmidt inner product.
Quantum Mechanics

Operator Growth Hypothesis

Thermalization

Chaos

Complexity

Hydrodynamics
Summary: arXiv: 1812.08657

- The hypothesis governs operator growth in chaotic, closed quantum systems

\[ b_n = \begin{cases} 
\frac{A_n}{W(n)} + O(1) \sim \frac{A_n}{\log n} & \text{if } d = 1 \\
\alpha n + \gamma + o(1) \sim \alpha n & \text{if } d \geq 2
\end{cases} \]

- Emergence of hydrodynamics in a computationally tractable scheme.

- The **operator growth rate** \( \alpha \) also controls the growth of complexity and chaos in quantum systems: \( \lambda_L \leq 2\alpha \)

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Future Work

- Log corrections disrupt asymptotics in 1D. How does our numerical technique still work in 1D?
- Can we prove the hypothesis within random matrix theory?
- How can we extend the hypothesis to finite temperature?
- Can we compute $b_n$ in QMC or other numerical techniques in 2D?
- Can we say anything about the MBL transition with this notion of chaos/ergodicity?
- Can we measure $\alpha$ experimentally? Perhaps from $\Phi(\omega)$ at large $\omega$?
Extra Slides
Translation-Invariant MBL (Preliminary)

Ising MBL model:

\[ H = \sum_j Z_j Z_{j+1} + 1.05X_j + W_j Z_j. \]

To recover translation-invariance, promote

\[ W_j \in \{-1, 1\} \rightarrow W_{\hat{\tau}_j^z} \]

where \( \hat{\tau}_j^z \) is a “binary disorder operator”.

\[ \tilde{H} = \sum_j Z_j Z_{j+1} + 1.05X_j + W_{\hat{\tau}_j^z} Z_j. \]

**Pro:** can compute Lanczos directly in the thermodynamic limit.

**Con:** doubled on-site operator space.
Translation-Invariant $ℓ$-bits (Preliminary)

Start with $\mathcal{O} = \mathbb{Z}$.

1. Truncate in Krylov space:

$$T(\Gamma) := \begin{pmatrix}
0 & b_1 & 0 & 0 & \cdots \\
b_1 & 0 & b_2 & 0 & \cdots \\
0 & b_2 & 0 & \cdots & \cdots \\
0 & 0 & \cdots & 0 & b_N \\
\vdots & \vdots & \vdots & b_N & -i\Gamma
\end{pmatrix}.$$ 

2. Solve

$$TA_\alpha = \varepsilon_\alpha A_\alpha$$

3. Smallest eigenvalue $\varepsilon_0(\Gamma) = 0 - iE_0\Gamma$

   with $A_0$ well-localized to small $n$.

Interpretation: $A_0$ is a “translation- and disorder-averaged $ℓ$-bit”.

![Graph showing the distribution of eigenvalues with different $\Gamma$ values and $\varepsilon_0$.]
Can we see the MBL transition?

For large-$W$, the “ℓ-bit” is exponentially localized in the chain.

For small-$W$, it decays as a power-law (fairly generic in ETH systems).

Interpretation: Lanczos gives a non-perturbative probe of the MBL transition.
History

Mathematical Results

- Araki (1969)
- Lieb-Robinson Bound (1972)
- ETH (1994)
- ADHH Theorem (2015)

![Diagram with axes x and t, and a shaded region representing $O(t)$]

OTOCs

- Quantum version of Lyapunov exponent (Kitaev)
- Computable in SYK, large-$N$, holography...
- Only well-defined semiclassically

Random Unitaries

- Solvable models of quantum chaos
- Local, finite-$N$, operator front propagation
- Emergent dissipation
- Non-Hamiltonian dynamics, no Lyapunov exponents
- (Nahum, Khemani, Huse, Pollmann, etc)
The Lanczos Algorithm

The Lanczos algorithm iteratively \textit{tridiagonalizes} a matrix

\textbf{Algorithm:}

1. Define

\[ |O_0 \rangle := O, \ b_0 := 0 \]

2. For each \( n \), apply \( \mathcal{L} \) to make a new operator:

\[ |A_n \rangle := \mathcal{L} |O_{n-1} \rangle - b_{n-1} |O_{n-2} \rangle \]

3. Orthogonalize again previous operator:

\[ |O_n \rangle := b_n^{-1} |A_n \rangle, \ b_n := (A_n |A_n \rangle)^{1/2} \]

4. Repeat until \( |O_n \rangle \) vanishes.

The Liouvillian becomes \textbf{tridiagonal}

\[ L_{nm} := (O_n |\mathcal{L}|O_m) = \begin{pmatrix} 0 & b_1 & 0 & 0 & \cdots \\ b_1 & 0 & b_2 & 0 & \cdots \\ 0 & b_2 & 0 & b_3 & \cdots \\ 0 & 0 & b_3 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \]

The \( b_n \)'s are called \textbf{Lanczos coefficients} and the \( |O_n \rangle \)'s are called \textbf{Krylov vectors}. 
Log Corrections in 1D

Theorem (Araki 1969) For any Hamiltonian with local interactions

\[ C(t + i\tau) = \langle \mathcal{O} | e^{i\mathcal{L}(t+i\tau)} | \mathcal{O} \rangle \]

is an entire function of \( t + i\tau \in \mathbb{C} \).

Corollary The asymptotic growth of the Lanczos coefficients is strictly sublinear in one dimension. In fact,

\[ b_n \leq A \frac{n}{W(n)} \]

where \( W \) is the product-log function defined by \( z = W(ze^z) \) whose asymptotic is \( W(n) \sim \ln n - \ln \ln n + O(1) \).

Therefore the hypothesis is modified in 1D. We still permit \( b_n \geq n^\alpha \) for any \( \alpha < 1 \).

1D correction via Alex Avdoshkin; Araki 1969; Abanin, De Roeck, Huveneers 2015.
Higher Dimensions

**Theorem (Bouch 2011)** For $d = 2$ (and higher), there exists a local Hamiltonian whose correlation function fail to be entire. Namely

$$H = \sum_{(x,y) \in \mathbb{Z}^2} Z_{x,y} X_{x+1,y} + X_{x,y} Z_{x,y-1}$$

with $\mathcal{O} = X_{0,0}$. (This achieves linear growth of $b_n$.)
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Corollary For $d \geq 2$, linear growth $b_n = \alpha n + O(1)$ is a tight upper bound for the growth of the Lanczos coefficients.

So the hypothesis survives unscathed in higher dimensions.

1D correction via Alex Avdoshkin & PhD thesis of G.D. Bouch, 2011
ETH Interpretation

Eigenstate thermalization hypothesis:

\[
O_{\alpha\beta} = O(\bar{E})\delta_{\alpha\beta} + e^{-S(\bar{E})} f_O(\bar{E}, \omega) R_{\alpha\beta}
\]  \hspace{1cm} (4)

where \(O\) is a local observable, \(\bar{E} = (E_\alpha + E_\beta)/2\), \(\omega = E_\alpha - E_\beta\), \(S(\bar{E})\) is the thermodynamic entropy, \(R_{\alpha\beta}\) a random variable and \(O(O)\) and \(f_O\) are smooth.

The operator growth hypothesis implies (at \(T = \infty\))

\[
\text{quantum chaos} \iff \int d\bar{E} \ f_O(\bar{E}, \omega) = e^{-\frac{\pi|\omega|}{2\alpha}} + O(1).
\]
Finite Temperature

At $T < \infty$ there is a physical choice of inner product. Suppose $g$ is an even measure on the thermal circle:

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